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Unbounded dynamics and compact invariant sets of one Hamiltonian system defined by the minimally coupled field

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ABSTRACT

In this paper we study some features of global dynamics for one Hamiltonian system arisen in cosmology which is formed by the minimally coupled field; this system was introduced by Maciejewski et al. in 2007. We establish that under some simple conditions imposed on parameters of this system all trajectories are unbounded in both of time directions. Further, we present other conditions for system parameters under which we localize the domain with unbounded dynamics; this domain is defined with help of bounds for values of the Hamiltonian level surface parameter. We describe the case when our system possesses periodic orbits which are found explicitly. In the rest of the cases we get some localization bounds for compact invariant sets.

1. Introduction

In this work we study the Hamiltonian system which describes a universe filled with a minimally coupled scalar field. This system is defined by a real Hamiltonian

$$H=\frac{1}{2}\left(-y_1^2+\frac{1}{x_1^2}y_2^2\right)-kx_1^2+\Lambda x_1^4+m^2x_1^4x_2^2+\frac{\omega^2}{x_1^2x_2^2} \qquad (1)$$

which has been derived in [14, see formula (3)] from the formula for the general action

$$\mathcal{I}=\frac{c^4}{16\pi G}\int\left[\mathcal{R}-2\Lambda-\frac{1}{2}\left(\nabla_\alpha\bar{\psi}\nabla^\alpha\psi+\frac{m^2}{\hbar^2}|\psi|^2\right)\right]\sqrt{-\mathfrak{g}}\,d^4x.$$

Here $\mathcal{R}$ is the Ricci scalar; Λ is the cosmological constant; ψ is the scalar field and m is so-called the mass of the field; $\mathfrak{g}$ is the determinant of the metric defined in (1) of the paper [14]. In what follows, it is assumed in (1) that real $m^2\neq 0$ and the phase is not constant, i.e. $\omega^2>0$. In accordance with [14] we may take parameter $k=1$ or -1. The most part of our paper concerns the case of positive Λ.

The nonexistence problem of periodic/homoclinic/heteroclinic orbits is studied for systems arisen in cosmology during many years. We note here papers of Wainwright and Hsu [20], Hewitt and Wainwright [7]. See also papers [2,4,6] and others. The main tool for nonexistence proofs of such objects is a construction of a differentiable function which is monotonic along positive half-trajectories.

Recently, this nonexistence problem has been considered as a part of a more general localization problem of compact invariant sets, see paper [8] in which the method based on first order extremum conditions has been described by Krishchenko. In works [12,13] the combined analytical–numerical approach to the hidden attractors localization has been presented by Leonov and his coauthors.

It is worth to mention that relevant results concerning localization/nonexistence of compact invariant sets have been reported in [9,16–19] for various cosmological systems.

We remind that finding the localization domain, i.e. domain containing all compact invariant sets is of essential interest for computer-based methods of its search and may provide some information respecting the long-time behavior of a system.

The stable interest to studies of chaotic systems with domains exhibiting unbounded dynamics is demonstrated during last three decades. Here we can mention only a short list of relevant papers containing analysis of global dynamics for the Rössler system in papers [11,1] and for open nonintegrable Hamiltonian systems in papers [5,10,3].

Systems with positive divergence of the flow in an invariant unbounded domain possess unbounded dynamics, i.e. escaping to infinity positive half-trajectories as a dominant behavior in this domain. In opposite to this, Hamiltonian systems are divergence-free, they preserve phase volumes and the problem of finding invariant domains with unbounded dynamics is nontrivial in this case. We notice that systems without compact invariant sets have no

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bounded motions and all trajectories are unbounded both in negative and in positive time. If a system possesses some unbounded invariant domain filled only by unbounded dynamics one may pose a localization problem of such domain as a localization problem of an invariant unbounded domain free from compact invariant sets.

In Hamiltonian systems with more than one degree of freedom their dynamics may substantially differs in different level surfaces. For example, we can meet a situation when in some level surfaces there are no compact invariant sets while in others they may exist. Therefore one may localize parameters defining Hamiltonian level surfaces which do not contain any compact invariant sets.

As it is known so far, Eqs. (2) are examined only in case $\omega = 0$. Here we recall that various integrability results for this system have been described in [14] by Maciejewski et al. In particular, it was established that the system (2) in the generic case is non-integrable. It is worth to mention also that integrability analysis of the Hamiltonian system for the conformally coupled scalar field has been presented in [15] also by Maciejewski et al.

This paper has three purposes.

1. We present a few groups of simple inequality-type conditions imposed on parameters of (1) and level surface parameter $l \in \mathbf{R}$; $H = l$, under which the system generated by the real Hamiltonian (1)

$$\begin{aligned}
\dot{x}_1 &= -y_1 \\
\dot{x}_2 &= \frac{1}{x_1^2} y_2 \\
\dot{y}_1 &= 2kx_1 - 4x_1^3(\Lambda + m^2x_2^2) + \frac{1}{x_1^3} y_2^2 + \frac{2\omega^2}{x_1^3 x_2^2} \\
\dot{y}_2 &= -2m^2x_1^4x_2 + \frac{2\omega^2}{x_1^2 x_2^3}
\end{aligned} \tag{2}$$

has no compact invariant sets in these level surfaces $H^{-1}(l)$. As a result, the system (2) contains only unbounded dynamics in indicated level surfaces.

2. We show how the localization method of compact invariant sets may provide explicit analytical formulae for loci of periodic orbits for (2); concerning another example one may consult in [9] where the authors have obtained explicit formulae for loci of heteroclinic orbits of the Raychaudhuri equations.

3. We provide a localization analysis for compact invariant sets.

This paper is organized as follows. Sections 2, 3 contain useful assertions (some preliminaries), respectively. In Section 4 we show that if $m^2 < 0$ or $m^2 > 0$; $\Lambda > 0$; $k = -1$ then the system (2) has no compact invariant sets. Further, in Section 4 we prove our main result: if $m^2 > 0$; $\Lambda > 0$; $k = 1$ and $l \le -\frac{1}{4\Lambda}$ or $l > \frac{1}{12\Lambda}$ then the system (2) has no compact invariant sets in level surfaces $H^{-1}(l)$. Next, we study in Section 5 the case $l = \frac{1}{12\Lambda}$. Namely, 1) if $\Lambda m^2\omega^2 > 2/27$ then there are no compact invariant sets in $H^{-1}(\frac{1}{12\Lambda})$; 2) if $\Lambda m^2\omega^2 = 2/27$ then the surface $H^{-1}(\frac{1}{12\Lambda})$ contains only four equilibrium points; 3) if $\Lambda m^2\omega^2 < 2/27$ then the surface $H^{-1}(\frac{1}{12\Lambda})$ contains four periodic orbits. In Section 6 we present some localization results for compact invariant sets concerning variables x_1, x_2 and y_2 in case $m^2 > 0$; $\Lambda > 0$; $k = 1$ and $-\frac{1}{4\Lambda} < l < \frac{1}{12\Lambda}$. Section 7 contains concluding remarks.

A few preliminary results of this work were presented by the author in PhysCon 2011 [19].

2. Useful background

Let us introduce a system

$$\dot{x} = F(x) \tag{3}$$

where $F(x)$ is a polynomial vector field on $\mathbf{R}^n$. By $\varphi(x,t)$ we denote a solution of (3), with $\varphi(x,0) = x$ for any $x \in \mathbf{R}^n$. Let $h(x)$ be a differentiable function such that h is not the first integral of (3). By $L_F h$ we denote the Lie derivative of the function h and by $S(h)$ we denote the set $\{x \in \mathbf{R}^n \mid L_F h(x) = 0\}$. For any set G in $\mathbf{R}^n$ we denote by $C\{G\}$ its complement and by $Cl\{G\}$ its closure. Further, let W be a subset in $\mathbf{R}^n$. We define $h_{\inf}(W) := \inf\{h(x) \mid x \in W\}$; $h_{\sup}(W) := \sup\{h(x) \mid x \in W\}$. The following results will be used in this paper:

Assertion 1. (See [8].) Let U be a domain in $\mathbf{R}^n$ and $M = S(h) \cap U$. Assume that $M \neq \emptyset$. Then each compact invariant set Γ of (3) contained in U is located in the set $K(U) = \{h_{\inf}(M) \le h(x) \le h_{\sup}(M)\}$ as well; $K(U)$ is called a localization set. If $M = \emptyset$ then the system (3) has no compact invariant sets contained in U.

Further, if the positive half-trajectory $\varphi(x,t)$, $t \ge 0$, is bounded then its ω-limit set, $\omega(x)$, is a non-empty compact invariant set. Its location is described by the classical LaSalle theorem which may be taken in the following form:

Assertion 2. Consider the system (3) and let $D \subset \mathbf{R}^n$ be an arbitrary set. Suppose that one can find a function V with continuous partial derivatives such that $L_F V \le 0$ on D. Let M be the maximal invariant set containing in the set $\{x \in Cl\{D\} \mid L_F V(x) = 0\}$. Let $\varphi(x,t)$ be a solution of (3) which remains in D for all $t \ge 0$. Then the solution $\varphi(x,t)$, with $t > 0$, approaches the set $M \cup \{\infty\}$ as $t \to \infty$. If M is bounded then either $\varphi(x,t) \to M$ or $\varphi(x,t) \to \infty$ as $t \to \infty$.

3. Some preliminaries and notations

The planes $x_i = 0$, $i = 1, 2$, brake the state space $\mathbf{R}^4 = \{(x_1, y_1, x_2, y_2)^T\}$ into four invariant domains

$$M_{++} = \{x_1 > 0; x_2 > 0\}; \qquad M_{+-} = \{x_1 > 0; x_2 < 0\};$$
$$M_{-+} = \{x_1 < 0; x_2 > 0\}; \qquad M_{--} = \{x_1 < 0; x_2 < 0\}.$$

Let $\Lambda > 0$. If $k = 1$ then our system possesses equilibrium points which are defined by equalities $y_1 = y_2 = 0$ and

$$x_{2*}^2 = \left| \frac{\omega}{m} x_{1*}^{-3} \right| \tag{4}$$

where x_{1*} is a real root of a cubic equation

$$2\Lambda|x_1^3| - |x_1| + |m\omega| = 0.$$

Further, if $\Lambda > 0$; $k = -1$ then (2) has no equilibrium points.

Below by $f(x_1, y_1, x_2, y_2)$ we denote the corresponding vector field of (2).

4. The nonexistence conditions of compact invariant sets

In this section we establish various nonexistence conditions of compact invariant sets which means unboundedness of global dynamics. We start from conditions which do not depend on parameter l.

Firstly we have

Proposition 1. *Let $m^2 < 0$. Then there are no compact invariant sets in $H^{-1}(l)$ for any l.*

Proof. Taking $h_1 = x_2y_2$ we have

$$L_f h_1 = -2m^2x_1^4x_2^2 + \frac{y_2^2}{x_1^2} + \frac{2\omega^2}{x_1^2x_2^2} > 0.$$