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Symplectic map description of Halley's comet dynamics

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ABSTRACT

We determine the two-dimensional symplectic map describing 1P/Halley chaotic dynamics. We compute the Solar system kick function, i.e. the energy transfer to 1P/Halley along one passage through the Solar system. Each planet contribution to the Solar system kick function appears to be the sum of a Keplerian potential and of a rotating gravitational dipole potential due to the Sun movement around Solar system barycenter. The Halley map gives a reliable description of comet dynamics on time scales of 10^4 yr while on a larger scales the parameters of the map are slowly changing due to slow oscillations of orbital momentum.

1. Introduction

The short-term regularity of 1P/Halley appearances in the Solar system (SS) contrasts with its long-term irregular and unpredictable orbital behavior governed by dynamical chaos [1]. Such chaotic trajectories can be described by a Kepler map [1,2] which is a two-dimensional area preserving map involving energy and time. The Kepler map was originally analytically derived in the framework of the two-dimensional restricted three body problem [2] and numerically constructed for the three-dimensional realistic case of 1P/Halley [1]. Then the Kepler map has been used to study nearly parabolic comets with perihelion beyond Jupiter orbital radius [2–5], 1P/Halley chaotic dynamics [1,6], mean motion resonances with primaries [7,8], chaotic diffusion of comet trajectories [7,9–12] and chaotic capture of dark matter by the SS and galaxies [13–15]. Alongside its application in celestial dynamics and astrophysics, the Kepler map has been also used to describe atomic physics phenomena such as microwave ionization of excited hydrogen atoms [16–18], and chaotic autoionization of molecular Rydberg states [19].

In this work we semi-analytically determine the symplectic map describing 1P/Halley dynamics, taking into account the Sun and the eight major planets of the SS. We use Melnikov integral (see, *e.g.* [4,20–24]) to compute exactly the kick functions associated to each major planet and in particular we retrieve the kick functions of Jupiter and Saturn which were already numerically extracted by Fourier analysis [1] from previously observed and computed 1P/Halley perihelion passages [25]. We show that each planet's contribution to the SS kick function can be split into a Keplerian potential term and a rotating dipole potential term due to the Sun movement around the SS barycenter. We illustrate the chaotic dynamics of 1P/Halley with the help of the symplectic Halley map and give an estimate of the 1P/Halley sojourn time. Then we discuss its long-term robustness comparing the semi-analytically computed SS kick function to the one we extract from an exact numerical integration of Newton's equation for Halley's comet orbiting the SS constituted by the eight planets and the Sun (see snapshots in Fig. 1) from −1000 to +1000 Jovian years around J2000.0, i.e. from about −10 000 BC to about 14 000 AD. Exact integration over a greater time interval does not provide exact ephemerides since Halley's comet dynamics is chaotic, see e.g. [6] where integration of the dynamics of SS constituted by the Sun, Jupiter and Saturn has been computed for 10^6 years.

2. Symplectic Halley map

Orbital elements of the current osculating orbit of 1P/Halley are [26]

$$e \simeq 0.9671, \quad q \simeq 0.586\,\text{au}, \quad i \simeq 162.3, \quad \Omega \simeq 58.42, \quad \omega \simeq 111.3, \quad T_0 \simeq 2446467.4\,\text{JD}$$

Along this trajectory (Fig. 1) the comet's energy per unit of mass is $E_0 = -1/2a = (e-1)/2q$ where a is the semi-major axis of the ellipse. In the following we set the gravitational constant $G = 1$, the total mass of the Solar system (SS) equal to 1, and the semi-major axis of Jupiter's trajectory equal to 1. In such units we have $q \simeq 0.1127$, $a \simeq 3.425$ and $E_0 \simeq -0.146$. Halley's comet pericenter

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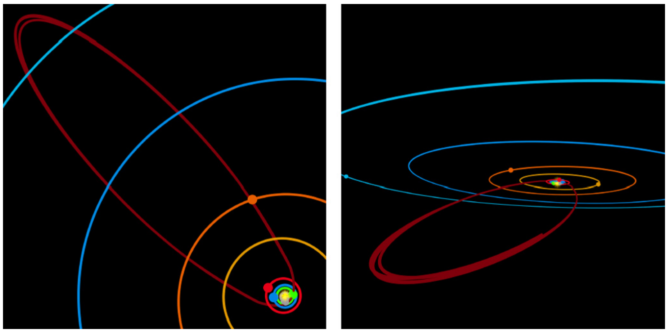

Fig. 1. Two examples of three-dimensional view of Halley's comet trajectory. The left panel presents an orthographic projection and the right panel presents an arbitrary point of view. The red trajectory shows three successive passages of Halley's comet through SS, the other near circular elliptic trajectories are for the eight Solar system planets, the yellow bright spot gives the Sun position. At this scale details of the Sun trajectory is not visible. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

can be written as $q = a(1-e) \simeq \ell^2/2$ where ℓ is the intensity per unit of mass of the comet angular momentum vector. Assuming that the latter changes sufficiently slowly in time we can consider the pericenter q as constant for many comet's passages through the SS. We have checked by direct integration of Newton's equations that this is actually the case ($\Delta q \simeq 0.07$) at least for a period of -1000 to $+1000$ Jovian years around J2000.0. Consequently, Halley's comet orbit can be reasonably characterized by its semi-major axis a or equivalently by Halley's comet energy E. During each passage through the SS many body interactions with the Sun and the planets modify the comet's energy. The successive changes in energy characterize Halley's comet dynamics.

Let us rescale the energy $w = -2E$ such as now positive energies ($w > 0$) correspond to elliptic orbits and negative energies ($w < 0$) to hyperbolic orbits. Let us characterize the nth passage at the pericenter by the phase $x_n = t_n/T_J \bmod 1$ where t_n is the date of the passage and T_J is Jupiter's orbital period considered as constant. Hence, x represents an unique position of Jupiter on its own trajectory. The energy w_{n+1} of the osculating orbit after the nth pericenter passage is given by

$$\begin{aligned} w_{n+1} &= w_n + F(x_n) \\ x_{n+1} &= x_n + w_{n+1}^{-3/2} \end{aligned} \tag{1}$$

where $F(x_n)$ is the kick function, i.e. the energy gained by the comet during the nth passage and depending on Jupiter phase x_n when the comet is at pericenter. The second row in (1) is the third Kepler's law giving the Jupiter's phase at the $(n+1)$th passage from the one at the nth passage and the energy of the $(n+1)$th osculating orbit.

The set of Eqs. (1) is a symplectic map which captures in a simple manner the main features of Halley's comet dynamics. This map has already been used by Chirikov and Vecheslavov [1] to study Halley's comet dynamics from previously observed or computed perihelion passages from -1403 BC to 1986 AD [25]. In [1] Jupiter's and Saturn's contributions to the kick function $F(x)$ had been extracted using Fourier analysis. In the next section we propose to semi-analytically compute the exact contributions of each of the eight SS planets and the Sun.

3. Solar system kick function

Let us assume a SS constituted by eight planets with masses $\{\mu_i\}_{i=1,\dots,8}$ and the Sun with mass $1-\mu = 1-\sum_{i=1}^{8}\mu_i$. The total mass of the SS is set to 1 and $\mu \ll 1$. In the barycentric reference frame we assume that the eight planets have nearly circular elliptical trajectories with semi-major axis a_i. We rank the planets such as $a_1 < a_2 < \ldots < a_8$ so a_5 and μ_5 are the orbit semi-major axis and the mass of Jupiter. The corresponding mean planet velocities $\{v_i\}_{i=1,\dots,8}$ are such as $v_i^2 = \left(1-\sum_{j\geq i}\mu_j\right)/a_i \simeq 1/a_i$. Here we have set the gravitational constant $G=1$ and in the following we will take the mean velocity of Jupiter $v_5 = 1$. The Sun trajectory in the barycentric reference frame is such as $(1-\mu)\,\mathbf{r}_\odot = -\sum_{i=1}^{8}\mu_i\mathbf{r}_i$.

In the barycentric reference frame, the potential experienced by the comet is consequently

$$\begin{aligned} \Phi(\mathbf{r}) &= -\frac{1-\mu}{\|\mathbf{r}-\mathbf{r}_\odot\|} - \sum_{i=1}^{8}\frac{\mu_i}{\|\mathbf{r}-\mathbf{r}_i\|} \\ &= \Phi_0(r)\left[1+\sum_{i=1}^{8}\mu_i\left(-1-\frac{\mathbf{r}\cdot\mathbf{r}_i}{r^2}+\frac{r}{\|\mathbf{r}-\mathbf{r}_i\|}\right)\right] \\ &\quad + o\left(\mu^2\right) \end{aligned} \tag{2}$$

where $\Phi_0(\mathbf{r}) = -1/r$ is the gravitational potential assuming all the mass is located at the barycenter.

Let us define a given osculating orbit $\mathcal{C}_0$ with energy E_0 and corresponding to the $\Phi_0(r)$ potential. The change of energy for the comet following the osculating orbit $\mathcal{C}_0$ under the influence of the SS potential $\Phi(\mathbf{r})$ (2) is given by the integral

$$\Delta E(x_1,\dots,x_8) = \oint_{\mathcal{C}_0}\nabla\left(\Phi_0(r)-\Phi(\mathbf{r})\right)\cdot d\mathbf{r} \tag{3}$$

which gives at the first order in μ

$$\begin{aligned} &\Delta E(x_1,\dots,x_8) \\ &\simeq \sum_{i=1}^{8}\mu_i\oint_{\mathcal{C}_0}\nabla\left(\frac{\mathbf{r}\cdot\mathbf{r}_i}{r^3}-\frac{1}{\|\mathbf{r}-\mathbf{r}_i\|}\right)\cdot d\mathbf{r} \\ &\simeq \sum_{i=1}^{8}\Delta E_i(x_i) \end{aligned} \tag{4}$$

This change in energy depends on the phases ($x_i = t/T_i \bmod 1$) of the planets when the comet passes through pericenter. From (4) we see that each planet contribution $\Delta E_i(x_i)$ are decoupled from the others and can be computed separately.

The integral (3) is similar to the Melnikov integral (see e.g. [4,20–24]) which is usually used in the vicinity of the separatrix to obtain the energy change of the pendulum perturbed by a periodic parametric term. In the case of the restricted 3-body problem the Melnikov integral can be used to obtain the energy change of the light body in the vicinity of 2-body parabolic orbit ($w \simeq 0$) [4]. We checked that integration (3) along an elliptical osculating orbit or along the parabolic orbit corresponding to the same pericenter give no noticeable difference as long as the comet semi-major axis is greater than planet semi-major axis. To be more realistic we adopt integration over an elliptical osculating orbit $\mathcal{C}_0$ since in the case of 1P/Halley slight differences start to appear for Neptune contribution to the kick function.

After the comet's passage at the pericenter, when the planet phases are $x_1,\dots,x_8$, the new osculating orbit corresponds to the energy $E_0 + \Delta E(x_1,\dots,x_8)$. Knowing the relative positions of the planets, the knowledge of e.g. $x = x_5$ is sufficient to determine all the x_i's. Hence, for Halley map (1) the kick function of the SS is $F(x) = -2\Delta E(x) = \sum_{i=1}^{8}F_i(x_i)$ where $F_i(x_i)$ is the kick function of the ith planet. In the following we present results obtained from the computation of the Melnikov integral (3) using coplanar circular trajectories for planets. We have checked the results are quite the same in the case of the non-coplanar nearly circular elliptic