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Bragg grating rogue wave

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ABSTRACT

We derive the rogue wave solution of the classical massive Thirring model, that describes nonlinear optical pulse propagation in Bragg gratings. Combining electromagnetically induced transparency with Bragg scattering four-wave mixing may lead to extreme waves at extremely low powers.

1. Introduction

Extreme wave phenomenon appears in a variety of scientific and social contexts, ranging from hydrodynamics and oceanography to geophysics, plasma physics, Bose–Einstein condensation (BEC), financial markets and nonlinear optics [1–5]. Historically, the first reported manifestation of extreme or rogue waves is the sudden appearance in the open sea of an isolated giant wave, with height and steepness much larger than the average values of ocean waves. A universal model for describing the dynamics of rogue wave generation in deep water with a flat bottom is the one-dimensional nonlinear Schrödinger (NLS) equation in the self-focusing regime. The mechanism leading to the appearance of NLS rogue waves requires nonlinear interaction and modulation instability (MI) of the continuous wave (CW) background [6]. Indeed, the nonlinear development of MI may be described by families of exact solutions such as the Akhmediev breathers [7], which are recognized as a paradigm for rogue wave shaping. A special member of this solution family is the famous Peregrine soliton [8], which describes a wave that appears from nowhere and disappears without a trace. Extreme waves that may be well represented by the Peregrine soliton have recently been experimentally observed in optical fibers [9], in water-wave tanks [10] and in plasmas [11].

Moving beyond the one-dimensional NLS model, it is important to consider extreme wave phenomenon in either multidimensional or multicomponent nonlinear propagation. Vector systems are characterized by the possibility of observing a coupling of energy among their different degrees of freedom, which substantially enriches the complexity of their rogue-wave families. Recent studies have unveiled the existence of extreme wave solutions in the vector NLS equation or Manakov system [12–15], the three-wave resonant interaction equations [16], the coupled Hirota equations [17] and the long-wave–short-wave resonance [18].

In this Letter, we present the rogue wave solution of the classical massive Thirring model (MTM) [19], a two-component nonlinear wave evolution model that is completely integrable by means of the inverse scattering transform method [20–22]. The classical MTM is a particular case of the coupled mode equations (CMEs) that describe pulse propagation in periodic or Bragg nonlinear optical media [23–27]. Furthermore, the CMEs also appear in other physical settings. In particular and relevant to rogue waves, they describe ocean waves in deep water for a periodic bottom [28]. As such, the search for novel solution forms of these equations including rogue waves, provides understanding of nonlinear phenomenon and leads to applications beyond optical systems. In this respect, benefiting from the result [25,29] that many MTM solutions (including single and multi-solitons and cnoidal-waves) may be mapped into solutions of the CMEs, provides a tool used in several works including ocean waves [30], BEC [31] and metamaterials [32].

After discussing the analytical rogue wave solution in Section 2, in Section 3 we numerically confirm its stability, and show that it may also be applied to describe the generation of extreme events

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in the more general context of the CMEs. Finally, in Section 4 we discuss the physical implementation of MTM rogue waves by using coherent effects in resonant nonlinear media, such as electromagnetically induced transparency (EIT), which may lead to the giant enhancement of cross-phase modulation (XPM) with the simultaneous suppression of self-phase modulation (SPM).

2. Analytical solution

Let us express the MTM equations for the forward and backward waves with envelopes U and V, respectively, as

$$U_\xi = -i\nu V - \frac{i}{\nu}|V|^2 U$$
$$V_\eta = -i\nu U - \frac{i}{\nu}|U|^2 V. \tag{1}$$

Here the light-cone coordinates ξ, η are related to the space coordinate z and time variable t by the relations $\partial_\xi = \partial_t + c\partial_z$ and $\partial_\eta = \partial_t - c\partial_z$, where $c > 0$ is the linear group velocity. Even though the arbitrary real parameter ν can be rescaled to unity, we find it convenient to keep it for dimensional reasons.

The rogue waves travel over the following CW background

$$U_0 = ae^{i\phi}, \qquad V_0 = -be^{i\phi} \tag{2}$$

where, with no loss of generality, the constant amplitudes a and b are real, and the common phase $\phi(\xi,\eta)$ is

$$\phi = \alpha\xi + \beta\eta, \qquad \alpha = b\left(\frac{\nu}{a} - \frac{b}{\nu}\right), \qquad \beta = a\left(\frac{\nu}{b} - \frac{a}{\nu}\right). \tag{3}$$

Up to this point we consider the two amplitudes a, b as free background parameters. It can be proved that rogue wave solutions of Eqs. (1) exist if and only if the two amplitudes a, b satisfy the inequality

$$0 < ab < \nu^2. \tag{4}$$

By applying the Darboux method to the MTM [33], one obtains the following rogue wave solution

$$U = ae^{i\phi}\frac{\mu^*}{\mu}\left(1 - 4i\frac{q_1^* q_2}{\mu^*}\right),$$
$$V = -be^{i\phi}\frac{\mu}{\mu^*}\left(1 - 4i\frac{q_1^* q_2}{\mu}\right) \tag{5}$$

with the following definitions

$$q_1 = \theta_1(1 + iq) + q\theta_2, \qquad q_2 = \theta_2(1 - iq) + q\theta_1,$$
$$q = \frac{a}{\chi^*}\eta + b\chi^*\xi \tag{6}$$

and

$$\mu = |q_1|^2 + |q_2|^2 + (i/p)\left(|q_1|^2 - |q_2|^2\right),$$
$$p = \sqrt{\frac{\nu^2}{ab} - 1} > 0. \tag{7}$$

In the expression (5), which is the analog of the Peregrine solution of the focusing NLS equation, the free parameters are the two real background parameters a, b, which are however constrained by the condition (4), and the two complex parameters θ_1, θ_2, while the parameter χ is given by the expression

$$\chi = \frac{b}{\nu}(1 + ip) = \frac{\nu}{a(1 - ip)}. \tag{8}$$

Expression (5) of the rogue wave solution may be simplified by fixing the reference frame of the space–time coordinates. The general solution (5) may then be obtained by applying to this particular solution a Lorentz transformation.

According to the last remark above, we now provide the rogue wave solution in terms of the space $z = c(\xi - \eta)$ and time $t = (\xi + \eta)$ coordinates directly. By rewriting the CW phase (3) in these coordinates, one obtains

$$\phi = kz - \omega t, \qquad k = \frac{\nu}{2c}\left(1 - \frac{ab}{\nu^2}\right)\left(\frac{b}{a} - \frac{a}{b}\right),$$
$$\omega = -\frac{\nu}{2}\left(1 - \frac{ab}{\nu^2}\right)\left(\frac{a}{b} + \frac{b}{a}\right), \tag{9}$$

where k is the wave number of the background CW. Setting $a = b$ means choosing the special frame of reference such that $k = 0$. Note that the other possibility $a = -b$ does not satisfy the condition that p is real (see (7)). From a physical standpoint, the CW background solution with $a = b$ corresponds to a nonlinear wave whose frequency $\omega = -\nu(1 - a^2/\nu^2)$ enters deeper inside the (linear) forbidden band-gap $\omega^2 < \nu^2$ as its intensity grows larger. A linear stability analysis of the CW background solution (2) shows that it is modulationally unstable for perturbations with a wavenumber $k^2 < 4a^2/c^2$ (for details, see [34]). Note that modulation instability gain extends all the way to arbitrarily long-scale perturbations (albeit with a vanishing gain), a condition which has been refereed to as "baseband instability", and that is closely linked with the existence condition of rogue waves in different nonlinear wave systems (e.g., the Manakov system, see Ref. [15]). It is also interesting to point out that, outside the range of existence of the rogue wave solution (5), that is for $a^2 > \nu^2$, the background is unstable with respect to CW perturbation with a finite (nonzero) gain (see Ref. [34]).

By using translation invariance to eliminate the parameters θ_1, θ_2, one finally ends up with the following expression of the MTM rogue wave solution

$$U = ae^{-i\omega t}\frac{\mu^*}{\mu}\left[1 - \frac{4}{\mu^*}q^*(q + i)\right],$$
$$V = -ae^{-i\omega t}\frac{\mu}{\mu^*}\left[1 - \frac{4}{\mu}q^*(q + i)\right] \tag{10}$$

where

$$\omega = -\nu\left(1 - \frac{a^2}{\nu^2}\right), \qquad q = -\frac{a^2}{\nu c}\left[ip(z - z_0) - c(t - t_0)\right],$$
$$p = \sqrt{\frac{\nu^2}{a^2} - 1}, \qquad \mu = 2|q|^2 + (1 + 2\,\mathrm{Im}\,q)\left(1 - \frac{i}{p}\right), \tag{11}$$

where z_0 and t_0 are arbitrary space and time shifts, respectively. Note that a further simplification may come from rescaling z, t, U, V by using the length scale factor $S = -\nu c/a^2$.

In Fig. 1 we show the dependence on space and time of the intensities $|U|^2$ and $|V|^2$ of the forward and backward components of the rogue wave (10). Here we have set $\nu = -1$, $c = 1$, $a = 0.9$, $t_0 = 2$ and $z_0 = 3.5$. As can be seen, the initial spatial modulation at $t = 0$ evolves into an isolated peak with a maximum intensity of about *nine times* larger than the CW background intensity. The corresponding contour plot of these intensities is shown in Fig. 2.

3. Numerical results

In order to verify the spatio-temporal stability of the rogue wave solution (10) over a finite spatial domain, we numerically solved Eqs. (1) with $\nu = -1$, $c = 1$, and using the initial (i.e., at $t = 0$) and boundary (i.e., at $z = 0$ and $z = L$) conditions given by