Physics Letters A 379 (2015) 888–893

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Linearization of the Lorenz system

Chunbiao Li [a,b,*], Julien Clinton Sprott [c], Wesley Thio [d]

[a] School of Electronic & Information Engineering, Nanjing University of Information Science & Technology, Nanjing 210044, China
[b] Engineering Technology Research and Development Center of Jiangsu Circulation Modernization Sensor Network, Jiangsu Institute of Commerce, Nanjing 211168, China
[c] Department of Physics, University of Wisconsin–Madison, Madison, WI 53706, USA
[d] Department of Electrical and Computer Engineering, The Ohio State University, Columbus, OH 43210, USA

ARTICLE INFO

Article history:
Received 24 November 2014
Received in revised form 6 January 2015
Accepted 7 January 2015
Available online 13 January 2015
Communicated by C.R. Doering

Keywords:
Piecewise linearization
Lorenz system
Amplitude control

ABSTRACT

A partial and complete piecewise linearized version of the Lorenz system is proposed. The linearized versions have an independent total amplitude control parameter. Additional further linearization leads naturally to a piecewise linear version of the diffusionless Lorenz system. A chaotic circuit with a single amplitude controller is then implemented using a new switch element, producing a chaotic oscillation that agrees with the numerical calculation for the piecewise linear diffusionless Lorenz system.

1. Introduction

The Lorenz model [1] describes the motion of a fluid under the conditions of Rayleigh–Bénard flow [2], and it has become a paradigm for chaotic dynamics. Furthermore, recent publications [3–10] show that the Lorenz system is still being actively researched. There is an inherent mechanism for the motion of the convective flow, which is governed by the stream function and the temperature deviation function. When the goal is to find the factors that lead to chaotic dynamics, it is necessary to consider the nonlinearity in the Lorenz model that represents a coupling between the fluid motion and the temperature deviation. The Lorenz equations provide a useful physical model of the dynamics assuming the actual fluid motion has only one spatial mode in the x direction and the temperature difference between top and bottom boundaries is not too large. Therefore, the Lorenz model has inherent limitations, and it is instructive to study diversified forms of it that could have physical implications. A natural question to ask is how the Lorenz system is modified when the amplitude information in the nonlinearity is removed by using a signum function, which leads to a piecewise linearization of the Lorenz model, which to our knowledge has not previously been done.

Furthermore, the piecewise linearity can be simply implemented electronically using diodes and operational amplifiers [11–16], whereas the usual quadratic nonlinearities require multipliers [17–19]. For some dynamical systems, this substitution preserves the chaotic dynamics. Another reason for doing this is that the resulting equations can be solved exactly in the linear regions with boundary conditions where the discontinuities occur. The method is analogous to Lozi's piecewise linearization of the Hénon map, where the quadratic term is replaced by an absolute-value term [20], or to the piecewise linearization of a jerk system by Linz and Sprott [21]. In addition, the piecewise linearization may allow a single amplitude control parameter [17,22,23], which is helpful for circuit implementation in radar or communication engineering to reduce the circuit complexity and avoid saturation of the amplifiers, which can be a problem because of the broadband frequency spectrum of a chaotic signal.

In this paper, linearization of the Lorenz system is achieved by ignoring the amplitude of one variable in the quadratic terms. What we are doing is not the same as the common linearization of a nonlinear system about an equilibrium point, but rather a piecewise linearization of a nonlinear system that retains the chaotic dynamics. In Section 2, one of the two quadratic terms is transformed into a non-smooth term with a signum operation, and a partially linearized version of the Lorenz system is derived. In Section 3, both of the quadratic terms are linearized by the signum operation, and a corresponding completely linearized version of the Lorenz system is obtained. Both cases have a total amplitude control parameter. In Section 4, a piecewise linear diffusionless

* Corresponding author at: School of Electronic & Information Engineering, Nanjing University of Information Science & Technology, Nanjing 210044, China. Tel./fax: +86 25 84685643.
E-mail addresses: goontry@126.com, chunbiaolee@gmail.com (C. Li).

http://dx.doi.org/10.1016/j.physleta.2015.01.003
0375-9601/

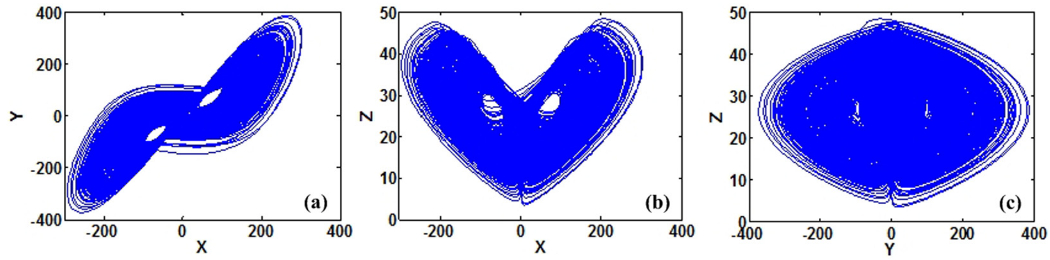

Fig. 1. Strange attractor from system (2) with $\sigma = 10$, $r = 28$, $\beta = 8/3$ for initial conditions $(0, 1, 0)$ with LEs $= (0.40_{56}, 0, -14.07_{23})$ (a) x–y plane, (b) x–z plane, (c) y–z plane.

Lorenz system is obtained by further simplification, which has the same structure as the quadratic case but with coexisting strange attractors for some values of the parameters. In Section 5, the bifurcation and multistability of the piecewise linear diffusionless Lorenz system is analyzed. The circuit implementation is presented in Section 6. Conclusions and discussions are given in the last section.

2. Partially linearized Lorenz system

The familiar Lorenz system is given by

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = -xz + rx - y$$
$$\dot{z} = xy - \beta z \tag{1}$$

with chaotic solutions for $\sigma = 10$, $r = 28$, $\beta = 8/3$. The system has rotational symmetry with respect to the z-axis as evidenced by its invariance under the coordinate transformation $(x, y, z) \rightarrow (-x, -y, z)$, and it has a partial amplitude control parameter hidden in the coefficient of the xy term, which controls the amplitude of x and y, but not z. To obtain total amplitude control, it is necessary to introduce an equal control factor into the xz term [22]. To obtain total amplitude control with a single parameter, it is necessary to make all the terms have the same order except for the one whose coefficient provides the amplitude control [17,23]. Since a signum operation will retain the polarity information while removing the amplitude information, applying it to one of the factors in a quadratic term reduces the order of that term from 2 to 1. Then the coefficient of the remaining quadratic term gives total amplitude control because it is the only term with an order different from unity. This idea leads to the partial linearization

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = -xz + rx - y$$
$$\dot{z} = x\,\mathrm{sgn}(y) - \beta z \tag{2}$$

With (σ, r, β) the same as for the quadratic system, system (2) gives the strange attractor shown in Fig. 1, which resembles the familiar Lorenz attractor, but with considerably larger x and y values. The Lyapunov exponents (LEs) given in the figure caption imply a Kaplan–Yorke dimension of 2.02_{88} and provide the main evidence that the system is chaotic. Note that the variable x appears in four of the seven terms, and thus it is especially important in determining the dynamic behavior. Although there are two ways to linearize xy, namely $x\,\mathrm{sgn}(y)$ and $y\,\mathrm{sgn}(x)$, it is reasonable that $x\,\mathrm{sgn}(y)$ works better for retaining the chaos. It is tempting to linearize the xz term in system (2) by replacing it with $x\,\mathrm{sgn}(z)$, but that destroys the coupling among the variables since z is always positive, and consequently, the first two dimensions will be independent of the third dimension.

Systems (1) and (2) both have three equilibrium points. The equilibrium points of system (1) are $(x, y, z) = (0, 0, 0)$ and $(\pm 8.4853, \pm 8.4853, 27)$, whose eigenvalues are $(11.8277, -2.6667, -22.8277)$ and $(-13.8546, 0.0940 \pm 10.1945i)$, respectively. The origin equilibrium point is a saddle-node, and the symmetric pair of equilibrium points are saddle-foci with identical eigenvalues. The equilibrium points of system (2) are $(0, 0, 0)$ and $(\pm 72, \pm 72, 27)$, whose eigenvalues are $(11.8277, -2.6667, -22.8277)$ and $(-14.9316, 0.6324 \pm 6.9152i)$, indicating the same stability as for system (1). For both systems, the rate of volume expansion is $-(\sigma + \beta + 1)$, and thus the systems are dissipative when the parameters are positive with solutions as time goes to infinity that contract onto an attractor of zero measure in their state space. However, the bifurcations for the parameters σ or β in the original system (1) and the revised system (2) are totally different. The revised system (2) shows relatively robust chaos over a range of both parameters. Specifically, there is a wide range of the parameter σ for system (2) to give a symmetric pair of coexisting strange attractors, while the original system (1) shows global attraction and bifurcations with different dynamics.

There is a well-known difficulty when calculating Lyapunov exponents for systems that involve discontinuous functions such as the signum. This problem arises because of the abrupt change in the direction of the flow vector at the discontinuity and the difficulty of maintaining the correct orientation of the Lyapunov vectors. Although there is a proper procedure for correcting this problem [24], we use here a simpler method in which $\mathrm{sgn}(y)$ is replaced by a smooth approximation given by $\tanh(Ny)$ with N sufficiently large that the calculated Lyapunov exponents are independent of its value [25]. For the case of system (2), a value of $N = 10$ is sufficient to give three-digit accuracy because of the large values of y. It is important with this method to use an integrator with an adaptive time step and error control to resolve the rapid change in the vicinity of $y = 0$ and to repeat the calculation with slightly perturbed initial conditions to verify the number of significant digits. Out of an abundance of caution, we quote only two significant digits in the largest Lyapunov exponents and include the remaining questionable digits as subscripts.

The linearized system (2) has two amplitude parameters, unlike system (1), which has only one. A new introduced coefficient h in the remaining quadratic term is a total amplitude controller,

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = -hxz + rx - y$$
$$\dot{z} = x\,\mathrm{sgn}(y) - \beta z \tag{3}$$

To show this, let $x = u/h$, $y = v/h$, $z = w/h$ to obtain new equations in the variables u, v, w that are identical to system (2). Therefore, the coefficient h controls the amplitude of all variables according to $1/h$. Otherwise, simply note that xz is the only term not of first order.

As with the quadratic system (1), a coefficient m in the signum term will realize partial amplitude control,