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Absolutely continuous invariant measure of a map from grazing-impact oscillators

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ABSTRACT

In this paper we investigate a one-dimensional map with unbounded derivative. The map is the limit of the Nordmark map which is the normal form of a discrete time representation of impact oscillators near grazing, i.e. when the dissipation of the systems is large, the Nordmark map can be viewed as a perturbation of the one-dimensional map. We prove that the map has an ergodic absolutely continuous invariant probability measure in a region of parameter space by constructing an induced Markov map.

1. Introduction and statement of the results

Impacting behavior exists in a large number of mechanical systems. Impacts may cause the mathematical models of the systems to have some non-smoothness, which gives rise to new forms of dynamics phenomena that are not found in smooth dynamical systems. A special situation arises when the impacts are with zero velocity, so-called grazing impacts. For impacts that are close to grazing, it is possible to condense the mathematical description into a discrete map as first done by Nordmark [20], followed by various authors [12,14,19,21,27,28]. Nordmark [20] derived a two-dimensional map

$$\begin{cases} x_{n+1} = \rho + \alpha x_n + y_n, \\ y_{n+1} = -\gamma x_n, \end{cases} \quad \text{if } x_n \le 0; \tag{1}$$

$$\begin{cases} x_{n+1} = \rho - \sqrt{x_n} + y_n, \\ y_{n+1} = -\gamma r^2 x_n, \end{cases} \quad \text{if } x_n > 0, \tag{2}$$

for an impact oscillator of one-degree-of-freedom which is driven by a periodic force in the neighborhood of a grazing state. For a harmonic oscillator of one-degree-of-freedom (see Fig. 1) whose motion equation is defined by

$$m\ddot{u} + \mu\dot{u} + ku = f\cos(\omega t),$$

the relationship between the Nordmark map and the physical system is as follows: (x_n, y_n) are the transformed coordinates in the position-velocity space $(u, \dot{u})$ of the impact oscillator evaluated at times $t_n = 2n\pi/\omega$. Eq. (1) governs the system if there is no impact between the time t_n and t_{n+1}, whereas (2) is applied when an impact takes place between the time t_n and t_{n+1}. The parameter α depends on intrinsic properties of the oscillator, γ is proportional to the friction coefficient, the restitution coefficient r gauges the energy loss at impact and ρ is proportional to the amplitude of the external force. The emergence of a square-root in Eq. (2) is characteristic for grazing collisions. A comprehensive study of the bifurcations of this map has been presented in the work of Chin et al. [7,8] and de Weger et al. [10,11].

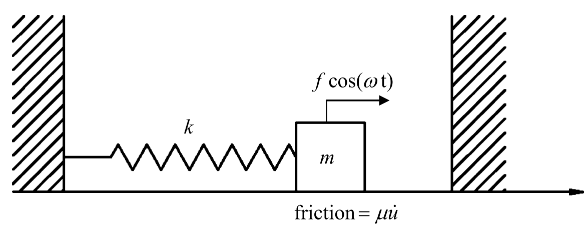

Fig. 1. Physical system modeled by the Nordmark map near grazing.

Except for transient behavior, we are sometimes interested in the evolution of the long time behavior of states of dynamical systems from a measure theoretic point of view. Invariant measures describe the relative frequency of certain parts of the phase space visited by typical orbits. We are interested particularly in those invariant measures which are relevant to physical observations. In the one-dimensional case, the invariant measures which are absolutely continuous with respect to the Lebesgue measures

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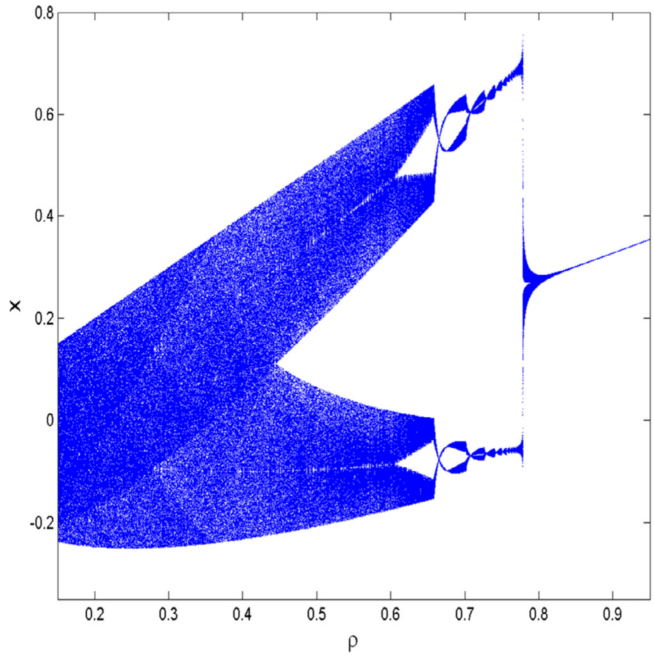
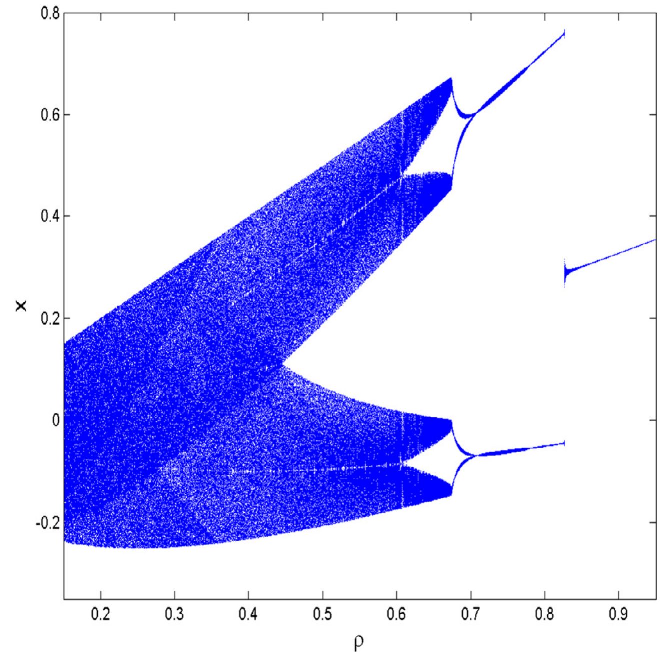


(a)



(b)

Fig. 2. (a) Bifurcation diagram of the map $f_{\alpha,\rho}(x)$ for $\alpha = 1.5$; (b) Bifurcation diagram of the Nordmark map for $\alpha = 1.5$, $\gamma = 0.001$ and $r = 1$.

satisfy such a property since the support of an absolutely continuous invariant measure is a set of positive Lebesgue measure and consequently Birkhoff's ergodic theorem describes the distributions of trajectories of points on a large set in the Lebesgue sense. On the other hand, absolutely continuous invariant measures are related to the chaotic behavior of dynamical systems, see for instance [5,17]. In the high-dimension case, SRB measures play an important role in the study of particular dissipative systems to exhibit chaotic behavior. SRB measures provide a mechanism for explaining how local instability on attractors can produce coherent stochastic-like behavior for orbits starting from large sets in the basin. We refer to [26] for more information on SRB measures.

When the dissipation is large, the Nordmark map can be viewed as a perturbation of the one-dimensional map

$$f_{\alpha,\rho}(x) = \begin{cases} \rho + \alpha x, & \text{if } x \le 0, \\ \rho - \sqrt{x}, & \text{if } x > 0, \end{cases} \qquad \alpha, \rho > 0,$$

see the following two bifurcation diagrams (Fig. 2). Avrutin et al. [1] studied how a square-root singularity influences the dynamical behavior for an extended version of the map. The aim of this paper is to prove that the map $f_{\alpha,\rho}(x)$ has an ergodic absolutely continuous invariant probability measure in a region of the parameter space. Although $f_{\alpha,\rho}(x)$ is expansive in the parameter region considered in this paper, we cannot apply the Lasota–Yorke theorem [16] here due to the existence of unbounded derivative. However, we can deal with this barrier by constructing an induced Markov map. Our main result is the following theorem.

Theorem 1. *There exists a set $\mathcal{A}$ of positive Lebesgue measure in the parameter space such that if $(a, \rho) \in \mathcal{A}$ then the map $f_{\alpha,\rho}$ has an ergodic absolutely continuous invariant probability measure on its invariant interval $[f^2(0), f(0)]$.*

We will define the set $\mathcal{A}$ in Section 3.

2. Preliminaries

In this section we recall the definitions of the Markov map and the induced Markov map and recall the condition for constructing an absolutely continuous invariant probability measure for the original map using its induced Markov map. Let $|I|$ denote the Lebesgue measure of an interval I.

Definition 2. A piecewise C^1 map $\varphi : I \to I$ on an interval I is called Markov if there exist a finite or countable collection $\{I_j\}$ of disjoint open intervals in I such that

(1) $I \backslash \cup_j I_j$ has the Lebesgue measure zero and if $\varphi(I_i) \cap I_j \neq \emptyset$ then $\varphi(I_i) \supset I_j$;
(2) there exists $\tau > 0$ such that $|\varphi(I_j)| \ge \tau$ for all j;
(3) there exists $K > 0$ and $\beta > 1$ such that $|D\varphi^n(x)| \ge K\beta^n$ for any $n \in \mathbb{Z}^+$ and $x \in I$ satisfying $\varphi^i(x) \in \cup_j I_j$ for all $0 \le i \le n$;
(4) there exist $C, \gamma > 0$ such that for each j and each $x, y \in I_j$, the following Hölder condition holds

$$\left| \frac{D\varphi(x)}{D\varphi(y)} - 1 \right| \le C \left| \varphi(x) - \varphi(y) \right|^{\gamma}.$$

It is well known that any interval map satisfying the above definition has an absolutely continuous invariant measure, see [9] for details. In the presence of critical points or singularities (in particular for singularities with unbounded derivative) the expansion or the bounded distortion condition in the above definition of the Markov map fails. However, in this setting, one can prove the existence of an absolutely continuous invariant measure for many maps of this type by associating an induced Markov map. Constructing induced Markov maps is a commonly used method to study the statistical properties of dynamical systems, especially for non-uniformly hyperbolic maps with critical points or singularities or co-existence of critical points and singularities, see for instances [3,4,6,13,15,18,22,24,25].

Definition 3. We say a map $\varphi : I \to I$ induces a Markov map if there is an interval $J \subset I$ and a Markov map Φ on J defined on a subset $\cup_j I_j$ of J with the full Lebesgue measure such that, for each j, the restriction of Φ to I_j is an iterate $\varphi^{k(j)}$ of φ with $\varphi^{k(j)}(I_j) \subset J$.