Physics Letters A 379 (2015) 518–523

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Maximal transport in the Lorenz equations

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ARTICLE INFO

Article history:
Received 19 July 2014
Received in revised form 15 October 2014
Accepted 22 October 2014
Available online 5 December 2014
Communicated by A.P. Fordy

Keywords:
Lorenz equations
Rayleigh–Bénard convection
Heat transport
Turbulence

ABSTRACT

We derive rigorous upper bounds on the transport $\langle XY \rangle$ where $\langle \cdot \rangle$ indicates time average, for solutions of the Lorenz equations without assuming statistical stationarity. The bounds are saturated by nontrivial steady (albeit often unstable) states, and hence they are sharp. Moreover, using an optimal control formulation we prove that no other flow protocol of the same strength, i.e., no other function of time $X(t)$ driving the $Y(t)$ and $Z(t)$ variables while satisfying the basic balance $\langle X^2 \rangle = \langle XY \rangle$, produces higher transport.

1. Introduction

Few mathematical models have had as profound an influence on the development of nonlinear science over the last half century as the Lorenz equations [1]

$$\dot{X} = -\sigma X + \sigma Y \tag{1}$$

$$\dot{Y} = rX - Y - XZ \tag{2}$$

$$\dot{Z} = XY - bZ. \tag{3}$$

This system arises as a severe modal truncation of Rayleigh's 1916 model of two-dimensional buoyancy-driven flow between parallel isothermal plates with stress-free boundaries [2]. In modern nondimensional variables Rayleigh's model is the Boussinesq approximation to the Navier–Stokes equations,

$$\dot{\omega} + J(\psi, \omega) = \sigma \Delta\omega + \sigma \,\mathrm{Ra}\theta_x \tag{4}$$

$$\dot{\theta} + J(\psi, \theta) = \Delta\theta + \psi_x \tag{5}$$

where the $J(\alpha, \beta) = \alpha_x\beta_y - \alpha_y\beta_x$, $\omega(x, y, t) = \Delta\psi(x, y, t)$ is the vorticity associated with steam function ψ, and $\theta(x, y, t)$ is the deviation of temperature from the steady linear conduction profile. The boundary conditions are $\psi = \psi_{yy} = \theta = 0$ at $y = 0$ and $y = 1$ with everything L-periodic in x. The dimensionless parameters of the problem are the Prandtl number σ, the ratio of diffusion of momentum to diffusion of heat in the fluid, and the Rayleigh number Ra, a ratio of the driving due to the temperature-drop-induced buoyancy force to the damping diffusion coefficients. Rayleigh proved that the steady conduction solution $\psi = 0 = \theta$ is linearly unstable to perturbations $\sim e^{ikx} \sin \pi y$ when $\mathrm{Ra} > \mathrm{Ra}_c(k) = (k^2 + \pi^2)^3/k^2$. The smallest critical Rayleigh number, $\frac{27}{4}\pi^4$, is achieved in domains of width $L = \text{integer} \times 2\sqrt{2}$.

Lorenz's variables are modal amplitudes in the Galerkin truncation approximation

$$\psi(x, y, t) = \frac{\sqrt{2}}{\pi}\left(\frac{k^2 + \pi^2}{k}\right) X(t) \sin kx \sin \pi y$$

$$\theta(x, y, t) = \frac{\sqrt{2}}{\pi r} Y(t) \cos kx \sin \pi y - Z(t) \frac{1}{\pi r} \sin 2\pi y \tag{6}$$

where the 'reduced' Rayleigh number $r = \mathrm{Ra}/\mathrm{Ra}_c$ and the domain-shape parameter $b = \frac{4\pi^2}{k^2+\pi^2}$. The time variable is also rescaled according to $t \to (k^2 + \pi^2)t$. Solutions of Rayleigh's continuum model are reasonably well approximated by Lorenz's truncation only near the primary bifurcation, i.e., for $r = \mathcal{O}(1)$, but the differential equations are nevertheless of theoretical (and historical) interest even for $r \gg 1$ due to the appearance of chaos in the solutions.

The bulk heat transport is gauged by the Nusselt number Nu, the ratio of the sum of the total (conductive plus convective) heat flux to the flow-independent conductive flux. The convective heat flux is proportional to the correlation between the vertical velocity ψ_x and the temperature θ, which reduces to $\mathrm{Nu} - 1 = \frac{k^2+\pi^2}{2\pi^2 r}\langle XY \rangle$

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http://dx.doi.org/10.1016/j.physleta.2014.10.050
0375-9601/

for Lorenz's variables where $\langle\cdot\rangle$ indicates the infinite time average (when the infinite time limit of long-but-finite time averages exist). The Nusselt number is a key indicator of the nonlinear response of the system to the driving whose strength is measured by the Rayleigh number (Ra or r). The classical linear and nonlinear stability results for both Rayleigh's and Lorenz's models are that the pure conduction state with $\mathrm{Nu}=1$, respectively $\psi=0=\theta$ and $X=Y=Z=0$, is absolutely stable for $\mathrm{Ra}<\mathrm{Ra}_c\equiv r<1$ and linearly unstable for $\mathrm{Ra}>\mathrm{Ra}_c\equiv r>1$.

It is of both fundamental theoretical interest and practical importance for applications to know the dependence of Nu on Ra, σ, and L. The high Rayleigh number Nu–Ra relationship characterizing turbulent convective heat transport is of interest for theory and experiment [3] and has remained the focus of mathematical analysis for more than half a century [4–6]. For Rayleigh's original 1916 model described above, for example, the most recent rigorous result is the upper bound $\mathrm{Nu}<.29\,\mathrm{Ra}^{5/12}$ uniformly in σ and L for $\mathrm{Ra}>\frac{27}{4}\pi^4$ [7].

The study of rigorous bounds on Nu for solutions of the Lorenz equations has received less attention with the notable exceptions of Malkus [8], Knobloch [9], and Foias et al. [10] who found that the steady state maximizes transport among statistically steady solutions and for invariant measures, and Pétrélis and Pétrélis [11] who proved that $\langle XY\rangle\le b\frac{(r+\sigma-\sqrt{\sigma})^2}{r+\sigma}$ for any solution. In this letter we present two alternative approaches to establish the improved estimate $\langle XY\rangle\le b(r-1)$, uniformly in σ for $r>1$, when the long-time limit exists. In case long-time averages do not converge, our result is that the limit supremum of finite-time averages satisfies the bound. Most significantly this upper bound is *sharp*: it is saturated by the exact steady solutions $(X_s,Y_s,Z_s)=(\pm\sqrt{b(r-1)},\pm\sqrt{b(r-1)},r-1)$.

In the next section we employ the so-called "background" method, originally contrived for estimating bulk averaged transport in solutions of the Navier–Stokes and related equations [5], to prove the new upper bound. The subsequent Section 3 introduces and develops a novel optimal control strategy for upper bound analysis to reproduce the result: we relax the momentum equation (1) and treat $X(t)$ as a control variable constrained only by $\langle X^2\rangle=\mathrm{Pe}^2$ to drive the temperature variables via (2) and (3). We prove in this setting that $\langle XY\rangle\le rb\,\mathrm{Pe}^2/(b+\mathrm{Pe}^2)$. Then auxiliary relation $\mathrm{Pe}^2=\langle XY\rangle$, from the neglected Eq. (1), can be used to connect the optimal transport with solutions of the Lorenz equations, yielding the same bound as obtained from the background analysis. This shows that no time-dependent stirring protocol, whether it solves the first Lorenz equation (1) or not, transports more than the steady flow. We also show, in a certain precise sense, that the steady stirring strategy is the *unique* maximizer.

2. Background analysis

We are interested in the $r>1$ parameter regime. It is convenient to rewrite the Lorenz equations as

$$\dot{x}=-\sigma x+\sigma r y \tag{7}$$

$$\dot{y}=x-y-xz \tag{8}$$

$$\dot{z}=xy-bz \tag{9}$$

where $X=x$, $Y=ry$ and $Z=rz$ and the Nusselt number in terms of the correlation of $x(t)$ and $y(t)$ is $\mathrm{Nu}=1+\frac{k^2+\pi^2}{2\pi^2}\langle xy\rangle$. (Note: do not confuse these lower case x and y variables with the spatial coordinates in Rayleigh's model discussed in the introduction.)

It is well known that, after possible initial transients, solutions of the Lorenz equations are uniformly bounded in time [12–17]. For example

$$\frac{1}{2}\frac{d}{dt}\left[\frac{1}{r^2}x^2+y^2+\left(z-1-\frac{\sigma}{r}\right)^2\right] = -\frac{\sigma}{r^2}x^2-y^2-bz^2+b\left(1+\frac{\sigma}{r}\right)z \tag{10}$$

so that

$$\overline{\lim_{t\to\infty}}\left[\frac{1}{r^2}x^2+y^2+\left(z-1-\frac{\sigma}{r}\right)^2\right] \le \begin{cases} (1+\frac{\sigma}{r})^2 & \text{if } \min\{1,\sigma,\frac{b}{2}\}=\frac{b}{2}\\ \frac{b^2(1+\frac{\sigma}{r})^2}{4(b-1)} & \text{if } \min\{1,\sigma,\frac{b}{2}\}=1\\ \frac{b^2(1+\frac{\sigma}{r})^2}{4\sigma(b-\sigma)} & \text{if } \min\{1,\sigma,\frac{b}{2}\}=\sigma. \end{cases} \tag{11}$$

Thus for differentiable functions $F:R^3\to R$, long time averages of time derivatives satisfy

$$\langle\dot{F}(x,y,z)\rangle_T\equiv T^{-1}\int_0^T\left[\frac{d}{dt}F\big(x(t),y(t),z(t)\big)\right]dt = \mathcal{O}\big(T^{-1}\big)\quad\text{as } T\to\infty. \tag{12}$$

Hence, averaging time derivatives of $\frac{1}{2}x^2$, $\frac{1}{2}(y^2+z^2)$, and $-z$ we deduce the balances

$$0=-\langle x^2\rangle_T+r\langle xy\rangle_T+\mathcal{O}\big(T^{-1}\big) \tag{13}$$

$$0=-\langle y^2\rangle_T-b\langle z^2\rangle_T+\langle xy\rangle_T+\mathcal{O}\big(T^{-1}\big) \tag{14}$$

$$0=-\langle xy\rangle_T+b\langle z\rangle_T+\mathcal{O}\big(T^{-1}\big). \tag{15}$$

Now write $z(t)=z_0+\varsigma(t)$ where, anticipating the result, we choose the time-independent "background" component $z_0=\frac{r-1}{r}$. Substituting this into (14) and (15) yields

$$0=-\langle y^2\rangle_T-b\langle\varsigma^2\rangle_T-2bz_0\langle\varsigma\rangle_T-bz_0^2+\langle xy\rangle_T+\mathcal{O}\big(T^{-1}\big) \tag{16}$$

$$0=bz_0+b\langle\varsigma\rangle_T-\langle xy\rangle_T+\mathcal{O}\big(T^{-1}\big). \tag{17}$$

Then the combination $(16)+2z_0\times(17)$ is

$$0=-\langle y^2\rangle_T-b\langle\varsigma^2\rangle_T+bz_0^2+(1-2z_0)\langle xy\rangle_T+\mathcal{O}\big(T^{-1}\big) \tag{18}$$

so that, adding zero cleverly disguised as $\frac{1}{r}\times(13)+r\times(18)$ to $(r-1)\langle xy\rangle_T$, we have

$$(r-1)\langle xy\rangle_T=rbz_0^2-\left\langle\left(\frac{x}{\sqrt{r}}-\sqrt{r}y\right)^2+rb\varsigma^2\right\rangle_T+\mathcal{O}\big(T^{-1}\big) \le rbz_0^2+\mathcal{O}\big(T^{-1}\big)=b\frac{(r-1)^2}{r}+\mathcal{O}\big(T^{-1}\big). \tag{19}$$

This, in turn, implies

$$\overline{\lim_{T\to\infty}}\,\langle XY\rangle_T=\overline{\lim_{T\to\infty}}\,r\langle xy\rangle_T\le b(r-1)=X_sY_s. \tag{20}$$

Therefore, when the long time limit exists, $\langle XY\rangle=\lim_{T\to\infty}\langle XY\rangle_T\le b(r-1)$ as advertised.

As a corollary it is interesting to note that the proof also shows that any sustained time dependence in the solutions, whether periodic or chaotic, strictly lowers the transport. Indeed, the first Lorenz equation (1) and the penultimate expression in (19) imply

$$\langle XY\rangle_T\le b(r-1)-\frac{1}{\sigma^2(r-1)}\langle\dot{X}^2\rangle_T+\mathcal{O}\big(T^{-1}\big) \tag{21}$$

so that $\langle XY\rangle$ is *strictly* less than X_sY_s when $\langle\dot{X}^2\rangle\ne0$.

This is illustrated in Fig. 1 where we plot the upper limit realized by the non-trivial steady state solutions along with measure-