Physics Letters A 379 (2015) 524–528

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Existence criterion of homoclinic trajectories in the Glukhovsky–Dolzhansky system

G.A. Leonov

St. Petersburg State University, Universitetsky pr. 28, St. Petersburg 198504, Russia

ARTICLE INFO

Article history:
Received 15 September 2014
Received in revised form 3 December 2014
Accepted 4 December 2014
Available online 9 December 2014
Communicated by C.R. Doering

Keywords:
Homoclinic trajectories
Tricomi problem
Fishing principle
Glukhovsky–Dolzhansky system
Phase space
Parameters space

ABSTRACT

Existence criterion of homoclinic trajectories in the Glukhovsky–Dolzhansky system, describing three-mode model of rotating fluid convection, is obtained. New applications of the Fishing principle are developed.

1. Introduction

In the papers [1–7], the methods of investigation of homoclinic trajectories were developed. These methods drawing on the classic ideas by Tricomi [8] enabled the development of criteria of existence along with numerical approaches to computing of the homoclinic trajectories. This development made it possible to consider the Lorenz, Shimizu–Morioka, Rössler, Lü, Chen, and Rabinovich systems [1–7]. The question arises as to whether it is possible to obtain the same results for the Glukhovsky–Dolzhansky system, describing three-mode model of rotating fluid convection [9]. This paper demonstrates that the difficulties, associated with resolving this question, can be overcome. To this end, new approaches to the investigation of nonlinear dynamical systems have been developed.

The Glukhovsky–Dolzhansky system describes a three-dimensional model of fluid convection inside the ellipsoid

$$\left(\frac{X_1}{a_1}\right)^2+\left(\frac{X_2}{a_2}\right)^2+\left(\frac{X_3}{a_3}\right)^2=1.$$

It is assumed that the ellipsoid rotates with the constant velocity Ω_0 around its axis a_3. The axis has a constant angle α with the gravity vector g. This vector is stationary with respect to the ellipsoid motion. The temperature difference is generated along the axis a_1 and a constant value q_0 is a gradient of this temperature. Here λ, μ, β are the coefficients of viscosity, heat conduction, and volume expansion, respectively.

Three-mode model of convection is obtained by Glukhovsky and Dolzhansky [9] in the following form

$$\begin{aligned}\dot{x}&=Ayz+Cz-\sigma x\\ \dot{y}&=-xz+Ra-y\\ \dot{z}&=-z+xy.\end{aligned}\qquad(1)$$

Here

$$\sigma=\lambda/\mu,\qquad Ta=\Omega_0^2/\lambda^2,$$

$$Ra=g\beta a_3 q_0/2a_1a_2\lambda\mu,$$

$$A=\frac{a_1^2-a_2^2}{a_1^2+a_2^2}(\cos\alpha)^2(Ta)^{-1}$$

$$C=\frac{2a_1^2a_2}{a_3(a_1^2+a_2^2)}\sigma\sin\alpha,$$

$$x=\mu^{-1}\omega_3,\qquad y=\frac{g\beta a_3}{2a_1a_2\lambda\mu}q_1,\qquad z=\frac{g\beta a_3}{2a_1a_2\lambda\mu}q_2,$$

$\omega_3(t)$ is the projection of the vector of angular velocity of fluid rotation on the axis a_3, $q_1(t)$ and $q_2(t)$ are the projections of temperature gradients on the axes a_1 and a_2, respectively. Here the other projections ω_1 and ω_2 are functions $\omega_1=-\frac{g\beta a_3}{2a_1a_2\Omega_0}\cos\alpha q_1$, $\omega_2=-\frac{g\beta a_3}{2a_1a_2\Omega_0}\cos\alpha q_2$, $q_3(t)\equiv 0$.

E-mail address: leonov@math.spbu.ru.

http://dx.doi.org/10.1016/j.physleta.2014.12.005

The parameters σ, Ta, and Ra are Prandtl, Taylor, and Rayleigh numbers, respectively.

After two sequential transformations:

$$x \to x, \quad y \to C^{-1}y, \quad z \to C^{-1}z,$$
$$x \to x, \quad y \to R - \frac{\sigma}{a_0 R + 1}z,$$
$$z \to \frac{\sigma}{a_0 R + 1}y,$$

one obtains the following system

$$\dot{x} = \sigma(y - x) - ayz$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = -z + xy. \tag{2}$$

Here $a_0 = A/C^2$, $R = Ra\,C$,

$$a = \frac{a_0\sigma^2}{(a_0R+1)^2}, \qquad r = \frac{R}{\sigma}(a_0R+1). \tag{3}$$

In what follows, for this system the existence criterion of homoclinic trajectories, similar to the criterion for the Lorenz system [1,2], will be developed.

Note that the Glukhovsky–Dolzhansky system is sufficiently different from the Lorenz system. In the Lorenz system, the flow of the two-dimensional convection is considered only. In the Glukhovsky–Dolzhansky system, the flow of the three-dimensional convection is considered which can be interpreted as one of the models of ocean flows [9].

2. Preliminaries

Consider a differential equation

$$\frac{dX}{dt} = f(X,q), \quad X \in R^n,\ q \in R^m, \tag{4}$$

where $f(X,q)$ is a smooth vector-function, $R^n = \{X\}$ is a phase space of system (4), $R^m = \{q\}$ is a parameter space of system (4).

Let $\gamma(s)$, $s \in [0,1]$ be a smooth path in the space of parameters $\{q\}$. Consider the following problem due to Tricomi [1–8]: Is there a point $q_0 \in \gamma(s)$ for which system (4) with q_0 has a homoclinic trajectory?

Recall that the trajectory $X(t)$ of system (4) is said to be homoclinic if the relation

$$\lim_{t\to+\infty} X(t) = \lim_{t\to-\infty} X(t) = X_0$$

is satisfied.

Consider system (4) with $q = \gamma(s)$, and let us introduce the following notion:

$X(t,s)^+$ is a separatrix of the saddle point X_0 ($\lim_{t\to-\infty} X(t,s)^+ = X_0$), with the one-dimensional unstable manifold, $X(s)^+$ is a point of the first crossing of a separatrix $X(t,s)^+$ with the closed set Ω:

$$X(t,s)^+ \,\overline{\in}\, \Omega, \quad t \in (-\infty, T),$$
$$X(T,s)^+ = X(s)^+ \in \Omega, \quad \text{Fig. 1}.$$

If such a crossing is lacking, then it is assumed that $X(s)^+ = \emptyset$. Here $\emptyset$ is an empty set.

Fishing principle. (See [1–7].) Suppose that for the path $\gamma(s)$ there is $(n-1)$-dimensional bounded manifold Ω with the piecewise-smooth edge $\partial\Omega$ that possesses the following properties:

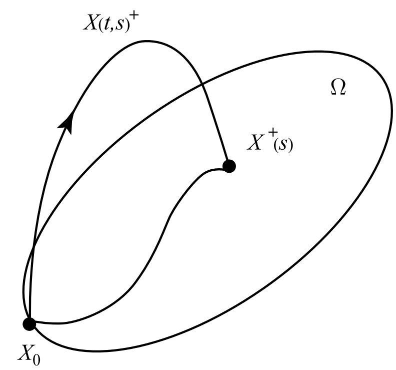

Fig. 1. Separatrix $X(t,s)^+$, $s \in [0, s_0]$.

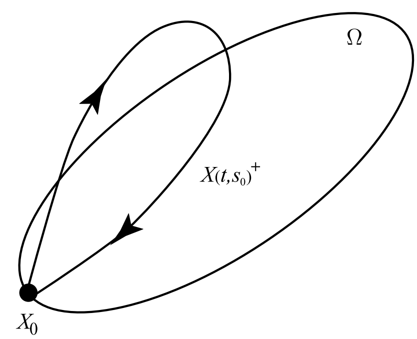

Fig. 2. Separatrix $X(t,s)^+$, $s = s_0$.

1) for any $X \in \Omega \setminus \partial\Omega$ and $s \in [0,1]$ the vector $f(X,\gamma(s))$ is transversal to the manifold Ω,
2) for any $s \in [0,1]$, $f(X_0,\gamma(s)) = 0$ and the point $X_0 \in \partial\Omega$ is a saddle of system (4),
3) the inclusion $X(0)^+ \in \Omega \setminus \partial\Omega$ is valid (Fig. 1),
4) the relation $X(1)^+ = \emptyset$ is satisfied,
5) for any $s \in [0,1]$ and $Y \in \partial\Omega \setminus X_0$ there exists a neighborhood $U(Y,\delta) = \{X \,|\, |X - Y| < \delta\}$ such that $X(s)^+ \,\overline{\in}\, U(Y,\delta)$.

Theorem 1. *(See [1–7].) If conditions 1)–5) are satisfied, then there exists $s_0 \in [0,1]$ such that $X(t,s_0)^+$ is a homoclinic trajectory of the saddle X_0 (Fig. 2).*

The Fishing principle can be interpreted as follows. In Fig. 1 is shown a fisherman at the point X_0 with the fishing rod $X(t,s)^+$. The manifold Ω is a lake surface and $\partial\Omega$ is a shore line.

When $s = 0$, a fish has been caught with the fishing rod. Then $X(t,s)^+, s \in [0,s_0)$ is the path of the fishing rod with the fish to the shore.

By assumption 5), the fish cannot be taken to the shore $\partial\Omega \backslash X_0$ since $\partial\Omega \backslash X_0$ is the forbidden zone.

Therefore, only the situation, shown in Fig. 2, is possible (i.e., at $s = s_0$, the fisherman has caught a fish). This corresponds to a homoclinic orbit.

Remark 1. In the papers [5,7], one more requirement was added to the Fishing principle:

6) for all s such that $X(s)^+ \in \Omega$ and for all $t \in (-\infty, T]$ there exists a number R such that $|X(t,s)^+| \le R$. Here $X(T,s)^+ = X(s)^+$.

However, this condition is always satisfied and consequently it is not included in Theorem 1.

Indeed, let us decompose the curve $X(s)^+$ into finite sequences of curves: