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On the distribution of local extrema in Quantum Chaos

Florian Pausinger [a], Stefan Steinerberger [b,*]

[a] IST Austria, Am Campus 1, 3400 Klosterneuburg, Austria
[b] Department of Mathematics, Yale University, 10 Hillhouse Avenue, New Haven, CT 06511, USA

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ABSTRACT

We numerically investigate the distribution of extrema of 'chaotic' Laplacian eigenfunctions on two-dimensional manifolds. Our contribution is two-fold: (a) we count extrema on grid graphs with a small number of randomly added edges and show the behavior to coincide with the 1957 prediction of Longuet-Higgins for the continuous case and (b) we compute the regularity of their spatial distribution using *discrepancy*, which is a classical measure from the theory of Monte Carlo integration. The first part suggests that grid graphs with randomly added edges should behave like two-dimensional surfaces with ergodic geodesic flow; in the second part we show that the extrema are more regularly distributed in space than the grid $\mathbb{Z}^2$.

1. Introduction

1.1. Quantum Chaos

Quantum Chaos is concerned with the behavior of high-frequency Laplacian eigenfunctions

$$-\Delta u = Eu \quad \text{on compact manifolds } (M, g)$$

and their seemingly chaotic properties. Apart from highly particular cases which are usually characterized by completely integrable behavior of the geodesic flow, these eigenfunctions will appear to be somewhat 'random'. Indeed, should the behavior be not chaotic, then usually any small perturbation of the geometry of the domain will induce chaotic behavior: randomness is the generic case. It is of great interest to try to understand this randomness by specifying arising invariants. See Fig. 1.

Some central questions of quantum chaos are

(1) whether (and under which conditions on the geometry of the manifold) the L^2-mass of the eigenfunctions tends towards uniform distribution – recent spectacular breakthroughs are due to Anantharaman [1] and Lindenstrauss [22].
(2) whether most eigenfunctions behave like 'random waves', i.e. whether for example

$$\frac{\|u_k\|_{L^\infty}}{\|u_k\|_{L^2}} \lesssim (\log k)^{\frac{1}{2}} \quad \text{with high probability [2]}$$

(3) how many nodal domains there are (see [4,5,7] for the random wave model and [6,30] for deterministic bounds) and how their volume is distributed (see e.g. [31]).

The number of nodal domains has received particular interest: in a highly influential paper by Blum, Gnutzmann and Smilansky [4], a universality statement for the number of nodal domains has been conjectured and numerically investigated: a generic Laplacian eigenfunction associated with the k-th eigenvalue seems to have $\sim 0.06k$ nodal domains. Bogomolny and Schmit [5] have worked out a percolation model simulating eigenfunctions in which the observation of Blum, Gnutzmann and Smilansky is confirmed: their model predicts that the number of nodal domains of the k-th eigenfunction is distributed with

$$\frac{3\sqrt{3}-5}{\pi}k \sim 0.06k \quad \text{mean and a variance of}$$

$$\left(\frac{18}{\pi^2}+\frac{4\sqrt{3}}{\pi}-\frac{25}{2\pi}\right)k \sim 0.05k.$$

It is not yet understood to what extent these numbers are precise outside the model (recent numerical work of Konrad [21] suggests the mean to be $\sim 4\%$ smaller); however, they are certainly very good approximations.

1.2. Chaotic eigenfunctions, local extrema and finite graphs

We are interested in the distribution of the local extrema of a Laplacian eigenfunction on a two-dimensional smooth surface with

* Corresponding author.
E-mail addresses: florian.pausinger@ist.ac.at (F. Pausinger), stefan.steinerberger@yale.edu (S. Steinerberger).

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0375-9601/

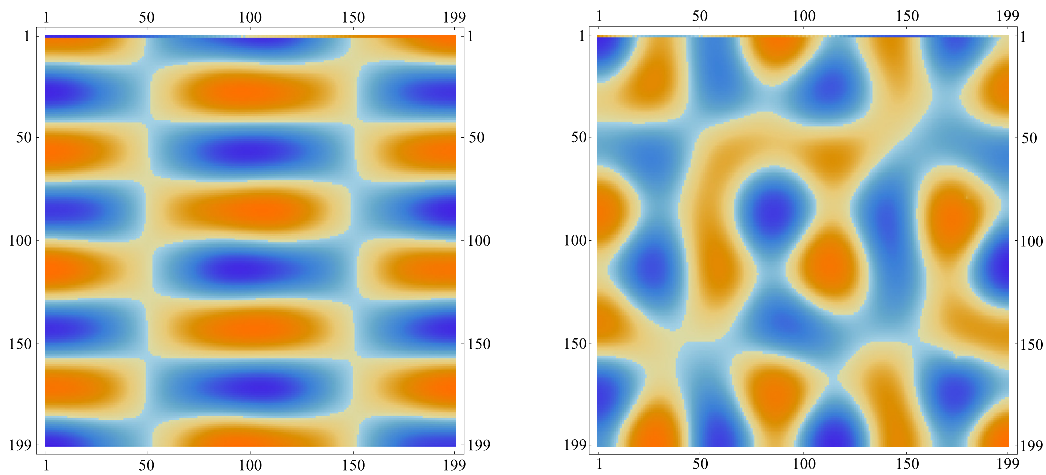

Fig. 1. An eigenfunction of the Laplacian with Neumann conditions once on $[0,1]^2$ (left) and once on a small perturbation of $[0,1]^2$ (right, the perturbation is not visible).

non-integrable geodesic flow. A cornerstone of existing conjectures is the random wave heuristic, which asserts that for all practical purposes a Laplacian eigenfunction should behave like a superposition of random plane waves

$$\psi(\mathbf{r}) = \sum_n a_k \cos\big(\langle \mathbf{k}_n, \mathbf{r}\rangle - \phi_n\big),$$

where a_k, ϕ_k are random reals and $\mathbf{k}$ is a randomly chosen direction normalized to $\|\mathbf{k}\| = \sqrt{E}$, where E is the energy/eigenvalue. Longuet-Higgins [23] studied this heuristic in a pioneering 1957 paper, which suggests that the n-th Laplacian eigenfunction on a compact two-dimensional surface should have $\sim n/\sqrt{3}$ extrema. The random wave approximation is of fundamental importance as its framework allows for precise computations while precise mathematical results seem still out of reach: for example, one would expect (see e.g. Yau [33]) that the nodal length has $(n-1)$-dimensional Hausdorff measure of size $\sim \sqrt{E}$ while the currently best rigorous results in dimensions ≥ 4 [8,28,29] do not even rule out the possibility that the nodal length might tend to 0 as $E \to \infty$.

It is natural to try simpler examples; a prime candidate is a reduction to finite graphs $G = (V, E)$. Given a finite, simple, connected Graph $G = (V, E)$ the natural analogue of the Laplacian is the discrete Graph-Laplacian (see e.g. [9]) given by a $|V| \times |V|$-matrix L with entries

$$L_{ij} = \begin{cases} 1 & \text{if } i = j \\ -(d_i d_j)^{-\frac{1}{2}} & \text{otherwise,} \end{cases}$$

where d_i is the degree of the vertex i. The first few eigenvalues/eigenvectors of the matrix will then approximate the first few eigenvalues/eigenfunctions of the Laplacian with Neumann boundary condition (for details we refer to [26]). It is not difficult to see that as the graph increases in size, it may be used to approximate the given geometry to any arbitrary degree of accuracy; however, counting nodal domains on graphs is rather difficult. Clearly, if two vertices u, v are joined by an edge e $u \sim_e v$ and the eigenfunction satisfies $f(u)f(v) < 0$, one would say that the edge crosses the nodal domain – the lack of continuity does not allow for an immediate transfer of the definition from the continuous case. Indeed, nodal domains of eigenfunctions on graphs are an ongoing field of great interest (see e.g. Davies, Gladwell, Leydold and Stadler [13], Dekel, Lee and Linial [14] or the survey [3]) but difficulty in transferring even very classical theorems from the continuous to the discrete setting poses a difficulty. In contrast, an extremum of a function is topologically simpler than that of a nodal domain and more easily generalized to the setting of a graph.

Contribution 1. Our first contribution is that grid graphs with a small number of randomly added edges behave like continuous surfaces with respect to the number of extrema of eigenfunctions and recover the Longuet-Higgins prediction. We also computed random wave approximations on both $\mathbb{T}^2$ and $\mathbb{S}^2$ to allow for comparison.

Using the fact that grid graphs with a small number of randomly added edges seem to provide a second way (the other being the random wave approximation) to create essentially 'chaotic' behavior, we use both ways to try to understand the way the extrema are distributed in space. We hasten to emphasize that graphs have, of course, been used by many people to describe chaotic behavior (see e.g. a paper of Smilansky [27], where d-regular graphs are employed); one possible advantage of using grid graphs with a random number of edges is their simplicity (the downside being, of course, that expander graphs, to give just one example, come with many additional properties which are not present in, say, the case of the grid graph.)

When studying the distribution of local extrema in space, we use discrepancy as a quantitative measure of regularity. Discrepancy is *the* standard measure in theory of uniform distribution (cf. classical books of Niederreiter [24] and Drmota and Tichy [17]) and has further applications in the theory of quasi-Monte Carlo integration. A point set with a small discrepancy is thus both well distributed from an abstract point of view as well as very suitable for numerical integration of a function with controlled oscillation.

Contribution 2. Extrema of chaotic Laplacian eigenfunctions are more regularly distributed with respect to discrepancy than the (suitably rescaled) classical grid $\mathbb{Z}^2$ (Fig. 2). In particular, they are better suited for numerical integration than the extrema of non-chaotic eigenfunctions (the extrema of the eigenfunction $\sin n\pi x \sin n\pi y$ of $-\Delta$ on $[0,1]^2$ are a translation of a rescaling of the grid $\mathbb{Z}^2$).

The rest of the paper is structured as follows: in Section 2 we describe our heuristic reasoning for why to employ grid graphs with randomly added edges to create quantum chaos, Section 3 shows that these graphs are able to reproduce the prediction of Longuet-Higgins on the number of local extrema, Section 4 introduces the discrepancy and describes the numerical results about the spatial distribution of Laplacian eigenfunction using both random wave models and the grid graphs with randomly added edges; technical comments and details about the implementation are given in the final section.