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Analytical method of spectra calculations in the Bargmann representation

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ABSTRACT

We formulate a universal method for solving an arbitrary quantum system which, in the Bargmann representation, is described by a system of linear equations with one independent variable, such as one- and multi-photon Rabi models, or N level systems interacting with a single mode of the electromagnetic field and their various generalizations. We explain three types of conditions that determine the spectrum and show their usage for two deformations of the Rabi model. We prove that the spectra of both models are just zeros of transcendental functions, which in one case are given explicitly in terms of confluent Heun functions.

1. Introduction

Our goal and result is a general method which allows to properly determine eigenvalues and eigenfunctions for a wide class of quantum systems. It is adequate for quantum optical setting where the Bargmann representation allows for natural parametrization of the electromagnetic degree of freedom and the resulting differential equations are ordinary and linear. We then show its application to two systems, which are generalizations of the famous Rabi model characterized by the Hamiltonian

$$H = a^\dagger a + \mu\sigma_z + \lambda\sigma_x\left(a^\dagger + a\right), \tag{1}$$

where a, $a^\dagger$ are the photon annihilation and creation operators, μ, λ are the level separation and photon–atom coupling constant, and σ_x, σ_z are the Pauli spin matrices.

This fundamental system describes interaction of a two-level atom with a single harmonic mode of the electromagnetic field. Originally, it was introduced to describe the effect of a rapidly varying, weak magnetic field on an oriented atom possessing nuclear spin [1]. It has been recently applied to a great variety of physical systems, including cavity and circuit quantum electrodynamics, quantum dots, polaronic physics and trapped ions, see [2–7].

Usually coupling between "natural" two-level atoms and the single bosonic mode of radiation is quite weak and the rotating wave approximation is valid. It leads to a solvable, the so-called Jaynes–Cummings, model. However, recent achievements in circuit quantum electrodynamics have enabled the exploration of such regimes, e.g., the ultra-strong and the deep strong coupling regimes of light–atom interaction so that the Jaynes–Cummings model begins to fail. Effects of counter-rotating terms cannot be more neglected and terms containing simultaneous excitation or deexcitation of both the atom and the field must be taken into account [2–6]. The second reason for its recent renaissance is the realization that the strong coupling regimes might require more interaction terms than just those mentioned above. One such generalization, the so-called Rabi model with broken symmetry, was proposed in [7] and its additional term was justified physically as spontaneous emission by the atom. The Hamiltonian of this generalization is

$$H_\varepsilon = a^\dagger a + \mu\sigma_z + \lambda\sigma_x\left(a^\dagger + a\right) + \varepsilon\sigma_x. \tag{2}$$

This will be the first example we study. The second one was proposed in [8,9]. It includes a nonlinear coupling term between the atom and the cavity:

$$H = \left(\omega + \frac{U}{2}\sigma_z\right)a^\dagger a + \frac{\omega_0}{2}\sigma_z + g\sigma_x\left(a^\dagger + a\right). \tag{3}$$

An alternative physical motivation of the additional term is that it could arise in the dispersive limit of the Jaynes–Cummings model.

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However, the first possibility is more accessible experimentally as described in [8]. We chose to keep the notation of that paper, for the second model, to facilitate comparison. A quick inspection shows that the parameters of models (2) and (3) are related via

$$\omega = 1, \qquad \omega_0 = 2\mu, \qquad g = \lambda. \tag{4}$$

Although the spectrum of the classical Rabi model has been determined by numerical and approximate methods before, see, e.g., [10–13], there still is a lack of a general approach which works well for arbitrary parameters values and which has a solid mathematical foundations. Recently several approaches devoted to determination of the spectrum of this and similar models have appeared, see, e.g., [7,14–17] and references therein. The authors have also applied the present method as outlined in the preliminary preprint [18], to determine the full spectrum in [19], including some isolated points that are usually overlooked.

It should be underlined that the Rabi model is one of the simplest ones in quantum physics. This is why the knowledge of its exact eigenvalues and eigenfunctions is of great theoretical importance. Although the question about the spectrum and eigenstates comes from physics, it is a mathematical one. It is obvious that unjustified methods may lead to incorrect physical interpretations of considered models.

In the Bargmann–Fock representation, see [20], the two-component wave function $\psi = (\psi_1, \psi_2)$ is an element of Hilbert space $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$, where $\mathcal{H}$ is the Bargmann–Fock Hilbert space of entire functions of one variable $z \in \mathbb{C}$. The elegant connection with the standard picture is that the annihilation and creation operators a, and $a^\dagger$ become ∂_z and multiplication by z, respectively, for clearly $[\partial_z, z] = 1$. The scalar product in $\mathcal{H}$ is given by

$$\langle f, g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} \overline{f(z)} g(z) e^{-|z|^2} \mathrm{d}\big(\mathrm{Re}(z)\big)\mathrm{d}\big(\mathrm{Im}(z)\big).$$

It is worth mentioning that this space was also introduced, independently of Bargmann, by J. Newman and H.S. Shapiro [21,22]. However, their motivation was connected with works of Ernst Fischer [23,24]. They tried to generalize a very beautiful construction of E. Fisher valid for polynomials.

The Hilbert space $\mathcal{H}$ has several peculiar properties. Let us mention two of them:

1. $f(z) \in \mathcal{H}$ does not imply that $f'(z) \in \mathcal{H}$.
2. $f(z) \in \mathcal{H}$ does not imply that $zf(z) \in \mathcal{H}$.

To understand these rather strange properties we have to recall some definitions and facts from the theory of entire functions, see [25,26]. If $f(z)$ is an entire function, then to characterize its growth, the following function is used:

$$M_f(r) := \max_{|z|=r} \big|f(z)\big|. \tag{5}$$

We omit the subscript f later on, because the investigated function is known from the context. If for an entire function $f(z)$ we have

$$\limsup_{r\to\infty} \frac{\ln(\ln M(r))}{\ln r} = \varrho, \quad \text{with } 0 \le \varrho \le \infty, \tag{6}$$

then ϱ is called the order (or growth order) of $f(z)$. If, further, the function has positive order $\varrho < \infty$ and satisfies

$$\limsup_{r\to\infty} \frac{\ln M(r)}{r^\varrho} = \sigma, \tag{7}$$

then we say that $f(z)$ is of order ϱ and of type σ.

Assume that $f(z)$ belongs to $\mathcal{H}$, then one can prove the following facts [20]:

1. $f(z)$ is of order $\varrho \le 2$.
2. If $\varrho = 2$, then $f(z)$ is of type $\sigma \le \frac{1}{2}$.

If $\varrho = 2$ and $\sigma = \frac{1}{2}$, then the question whether $f(z) \in \mathcal{H}$ requires a separate investigation. Particularly in the mentioned case when $f(z) \in \mathcal{H}$ but $f'(z) \notin \mathcal{H}$ the function is of order $\varrho = 2$ and type $\sigma = \frac{1}{2}$. For additional details see [27].

The usefulness of this representation can immediately be seen with the harmonic oscillator, which represents the radiation. The time-independent Schrödinger equation for energy E is simply $H\psi(z) = z\psi'(z) = E\psi(z)$ and one immediately recovers the orthonormal eigenbasis as $\{z^n/\sqrt{n!}\}_{n\in\mathbb{N}}$. The connection with the usual space of square-integrable functions of q is given by the integral kernel $\exp(-(z^2+q^2)/2 + \sqrt{2}qz)$ which is one of the forms of the generating function for the Hermite polynomials. Each z^n thus corresponds to the appropriately normalized wave function $e^{-q^2/2}H_n(q)$. In this basis the operator a is just an infinite matrix with entries on the superdiagonal, so all the mentioned Hamiltonians can be constructed as tensor products of such matrices with the sigma matrices. This allows for direct numerical diagonalization. However, the open question that we wish to tackle is how to determine the spectrum rigorously with as explicit exact formulas as possible.

In the Bargmann–Fock representation energy E belongs to the spectrum of the problem, if and only if, for this value of E the equation $H\psi = E\psi$ has entire solution $\psi = (\psi_1, \psi_2) \in \mathcal{H}^2$. We want to find, if possible, a formula for those values of E.

As we already mentioned, in the Bargmann–Fock representation, the considered models are described by a system of linear differential equations. We shall see that the equations in question will involve regular singular points and a possibly irregular point at infinity on the complex z plane. The conditions that the considered system has a solution with components belonging to $\mathcal{H}$, are roughly threefold:

- Local conditions. At each regular singular point $z = s$ there exists at least one solution which is holomorphic on an open set containing s.
- Global conditions. Among all solutions which are locally holomorphic, we can find at least one at each singular point such that they are a holomorphic continuation of one another.
- Normalization conditions. The entire function obtained in the above way must have finite Bargmann norm.

Our method gives straightforward and natural compatibility conditions in term of Wronskian determinants and was first formulated in our unpublished preprint [18]. For simplicity sake we chose the two models that can be given either as a system of two first order equations or one equation of the second order. The application to higher order equations, as those investigated in [28] or [29], will appear in future work [30].

In the Bargmann representation, the first considered model is described by the following system of two differential equations

$$\begin{aligned}(z+\lambda)\frac{\mathrm{d}\psi_1}{\mathrm{d}z} &= (E-\varepsilon-\lambda z)\psi_1 - \mu\psi_2,\\ (z-\lambda)\frac{\mathrm{d}\psi_2}{\mathrm{d}z} &= (E+\varepsilon+\lambda z)\psi_2 - \mu\psi_1.\end{aligned} \tag{8}$$

We will use this model to illustrate the single equation approach below. The second system takes the form

$$\begin{aligned}\left(\omega+\frac{U}{2}\right)z\psi_1' + \frac{\omega_0}{2}\psi_1 + g\psi_2' + gz\psi_2 &= E\psi_1,\\ \left(\omega-\frac{U}{2}\right)z\psi_2' - \frac{\omega_0}{2}\psi_2 + g\psi_1' + gz\psi_1 &= E\psi_2.\end{aligned} \tag{9}$$