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Irregular mixing due to a vortex pair interacting with a fixed vortex

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ABSTRACT

The paper examines scalar advection caused by a point–vortex pair encountering a fixed point vortex in a uniform flow. The interaction produces two types of vortex motion. First is unbounded as the pair moves unrestrictedly after encountering the fixed vortex. The scalar exchanging between the pair's bubble and fixed vortex's neighbourhood is numerically estimated. Second is bounded as the pair's vortices periodically oscillate about the fixed vortex. The pair's periodic motion perturbs scalar motion causing a portion of scalar trajectories to manifest chaotic behaviour. We analyse scalar transport using Poincaré sections, which reveal regular and chaotic transport regions.

1. Introduction

Passive scalar advection due to point vortices is a long studied problem that has a multitude of applications in fluid mechanics. For instance, a great number of vortices are used as low-dimensional models of turbulence [1–7], while a few vortices are often considered as the simplest models exhibiting irregular (chaotic) dynamics. Without any constraints, point–vortex systems become irregular when there are more than three vortices [8–12]. However, it is the three-vortex case when the passive scalar advection turns chaotic [13,14]. With certain symmetric configuration, the four-vortex problem is integrable but also causes chaotic passive scalar advection [15]. We use the term chaotic implying that two initially close phase trajectories in the corresponding phase space tend to diverge exponentially in a finite time while no stochastic impact affects the system [16,17]. Thus our system is presumed to be fully deterministic.

However, if one is interested in a configuration when additional constraints are superimposed to a point–vortex system, the irregular dynamics can manifest itself at a smaller number of point vortices. For instance, if a background shear flow is superimposed to a point–vortex system, just three vortices start moving chaotically [18], while passive scalar advection turns chaotic due to just two point vortices [19–23].

In general, different constraints are superimposed to model certain physical processes. For instance, it turned out, that the point–vortex systems can give a lot of qualitative information to facilitate studying the mesoscale dynamics of oceanic vortices [24–32] when the low-scale dynamics can be neglected. And this is in effect when one considers the interacting vortices as coherent structures, which remain rather intact for a given time. Thus, if one can distinguish the interacting vortices rather clearly during all the interacting time, then the point–vortex model may provide with some insight into the processes under investigation. The possibility of distinguishing coherent structures for a long time is rather strong in the ocean as it is generally quasi-two dimensional, which allows one to exploit the quasi-geostrophic approximation [33].

A number of works consider a fixed point vortex as the simplest model of topography that induces a closed recirculation zone in its vicinity [34–36]. Given any background flow, such a closed recirculation zone, which may be thought of as a topographically trapped vortex, gets enveloped by a material line called the separatrix that divides distinct regions of the flow. Such models can be useful for instance to study the vortex dynamics associated with vortex and wave trapping by topography. This work addresses the point–vortex model of a similar interaction, namely the dynamics of a vortex pair encountering isolated topography. All the vortices in the model are assumed to be point vortices. It is established that this configuration produces two types of the vortex pair motion [37,38]. First is an unbounded propagation, when the pair moves to infinity in an almost rectilinear path after being deflected by the topography, and second is a bounded regime, when the pair oscillates periodically about the topography. The latter is of main interest in this study since it provides a periodically perturbed dynamical system that manifests the chaotic dynamics. Thus, numerical analysis of scalar transport in both unbounded and periodic motion regimes is the main goal of this work.

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2. The dynamics of a point–vortex dipole encountering a fixed point vortex

To begin, we review the dynamics of a point–vortex dipole encountering a fixed point vortex. We assume the pair to be a structure, that consists of two counter-rotating point vortices with equal strengths μ. This pair is directed at a fixed point vortex with strength σ, that plays the role of isolated topography. The stream-function of the flow, then, has a Hamiltonian form

$$\psi = \sigma \ln(r_0) + \mu \ln\left(\frac{r_1}{r_2}\right), \tag{1}$$

where $r_0 = ((x-x_0)^2 + (y-y_0)^2)^{1/2}$ is the distance from a fluid particle (x, y) to the fixed vortex (x_0, y_0), $r_1 = ((x-x_1)^2 + (y-y_1)^2)^{1/2}$, $r_2 = ((x-x_2)^2 + (y-y_2)^2)^{1/2}$ are the analogous distances to the corresponding vortex of the pair. Then, taking into account that variables (x, y) are canonical, the governing equations for the motion of a fluid particle ensue,

$$\begin{aligned}\frac{dx}{dt} &= -\frac{\partial\psi}{\partial y} = -\left(\sigma\frac{(y-y_0)}{r_0^2} + \mu\left(\frac{(y-y_1)}{r_1^2} - \frac{(y-y_2)}{r_2^2}\right)\right),\\ \frac{dy}{dt} &= \frac{\partial\psi}{\partial x} = \sigma\frac{(x-x_0)}{r_0^2} + \mu\left(\frac{(x-x_1)}{r_1^2} - \frac{(x-x_2)}{r_2^2}\right).\end{aligned} \tag{2}$$

To obtain governing equations for the motion of the vortices, one needs to introduce corresponding vortex coordinates (x_1, y_1) or (x_2, y_2) instead of variables (x, y) in (2), and, then, to omit the singularities caused by the self-action of a vortex on itself. The governing equations for the vortex trajectories thus are

$$\begin{aligned}\frac{dx_i}{dt} &= -\sigma\frac{(y_i-y_0)}{r_{0i}^2} + \mu\frac{(y_i-y_j)}{r_{ij}^2},\\ \frac{dy_i}{dt} &= \sigma\frac{(x_i-x_0)}{r_{0i}^2} - \mu\frac{(x_i-x_j)}{r_{ij}^2},\end{aligned} \tag{3}$$

where $i = 1, 2$, $j = 1, 2$, $i \neq j$, and $r_{0i} = ((x_i-x_0)^2 + (y_i-y_0)^2)^{1/2}$, $r_{ij} = ((x_i-x_j)^2 + (y_i-y_j)^2)^{1/2}$.

Although system (3) allows significant simplifications, in the general case, it still cannot be integrated in quadratures, and therefore should be treated numerically. It is established [38] that, in accord with system (3), the pair can perform two distinct types of motion. First is an unbounded motion, when the pair moves towards the fixed vortex, then, interacts with it, but after the interaction it propels itself further to infinity in an almost rectilinear way. Second is a bounded periodic motion, when the pair fluctuates about the fixed vortex. As to particle advection, both regimes are somehow interesting since in the unbounded regime the pair interacts with the fixed vortex only for a finite time, and during this time, a portion of particles initially located inside the pair's bubble, a fluid region the pair drags along as it self-propagates, can leave it while particles from the vicinity of fixed vortex replace them and are then carried off the fixed vortex. Hence, exchange of particles between the pair's bubble and the fixed vortex's vicinity occurs. The second regime, on the contrary, is more intricate as the periodic motion of the pair plays the role of a periodic perturbation to fluid particles resulting in chaotic advection, an exponential divergence of initially close particles in a finite time. Examples of the vortices' trajectories are plotted in Fig. 1. Figs. 1a, and 1b correspond to the unbounded motion case as $\sigma = 30$, $\mu = 10$, $x_1(0) = 30.1$, $y_1(0) = 0$, $x_2(0) = -15$, $y_2(0) = 0$, and the bounded motion case as $\sigma = 30$, $\mu = 10$, $x_1(0) = 30.1$, $y_1(0) = 0$, $x_2(0) = -14.19178$, $y_2(0) = 0$, respectively.

First, we look into the unbounded motion case, when the vortex pair after interacting with the fixed vortex advances to infinity. It was shown [38], that system (3) can be integrated in elementary functions if the pair's vortices are located symmetrically relating to the fixed vortex. Let us represent system (3) in polar coordinates $r^2 = x^2 + y^2$, and $\theta = \arctan\frac{y}{x}$,

$$\begin{aligned}\frac{dr_1}{dt} &= \frac{\mu r_2 \sin(\theta_1-\theta_2)}{r_{12}^2},\\ \frac{dr_2}{dt} &= \frac{\mu r_1 \sin(\theta_1-\theta_2)}{r_{12}^2},\\ \frac{d\theta_1}{dt} &= -\mu\frac{r_1 - r_2\cos(\theta_1-\theta_2)}{r_1 r_{12}^2} + \frac{\sigma}{r_1^2},\\ \frac{d\theta_2}{dt} &= \mu\frac{r_2 - r_1\cos(\theta_1-\theta_2)}{r_2 r_{12}^2} + \frac{\sigma}{r_2^2}.\end{aligned} \tag{4}$$

Then, the first two relations yield that angular momentum $M = \mu({r_1}^2 - {r_2}^2) \equiv const$. Thus, supposing symmetrical initial distribution of the pair's vortices relating to the fixed vortex $r_1(0) = r_2(0)$ then $M = 0$, i.e. $r_1 = r_2$ for every instant in time. Hence, the increasing solution ensues

$$\begin{aligned}r_1(t) &= \frac{r_{12}}{2}\left(1 + \Phi^2(t)\right)^{1/2},\\ \theta_1(t) &= \frac{2\sigma-\mu}{\mu}\left(\arctan\Phi(t) - \arctan\left(4\left(\frac{r_1(0)}{r_{12}}\right)^2 - 1\right)^{1/2}\right),\end{aligned} \tag{5}$$

where $\Phi(t) = \frac{2\mu}{r_{12}^2}t - (4(\frac{r_1(0)}{r_{12}})^2 - 1)^{1/2}$, and r_{12} is the constant distance between the pair's vortices. From (5), it is clearly seen that, for every symmetric initial distribution of the pair's vortices relating to the fixed vortex, the pair moves to infinity at a constant angle provided $2\sigma = \mu$.

The bounded motion case is periodical in a reference frame rotating with a constant angular velocity. This angular velocity, which depends on strengths σ, μ, and the vortices' initial positions, can be figured out by taking advantage of the fact, that the system rotates over constant angle Θ in one period T of the radii r_1, r_2 changing. Transiting to the reference frame rotating with angular velocity $\Omega = \Theta/T$, we obtain the periodic phase trajectories like those shown in Fig. 1c, which are plotted for the same values of the parameters as in Fig. 1b. Considering the pair motion in the rotating reference frame, it is clear that it is periodic as the trajectories form closed loops. So, further we will study how the periodic motion of the pair affects fluid particle advection occurring in the vicinity of the dipole–topography interaction.

3. Passive scalar advection

In the case of the unbounded propagation, a useful quantity to assess advection properties is the number of the particles that have initially been located inside the pair's bubble, and then, after the vortex pair interacts with the fixed vortex, are stayed entrapped in the vicinity of the fixed vortex. Thus, if the number of particles varies, one can compare the transport properties of the flow for different initial conditions.

First, one needs to specify an initial distribution of markers that will be followed as the flow unfolds. To that end, we first recall the dynamics of a self-propelling pair in a uniform unbounded fluid [39], which corresponds to the problem under study when the pair is infinitely far from the fixed vortex so its influence becomes negligible. Since in this case all the directions are equivalent, let the pair move along the x-axis, then it moves rectilinearly and uniformly with velocity $U = -\frac{\mu}{2y_0}$, where $(0, y_0)$, $(0, -y_0)$ are the initial positions of the vortex with strength μ and $-\mu$ respectively. The governing equations for the vortices are simply $x_1 = x_2 = Ut$,