Physics Letters A 377 (2013) 1065–1069

Contents lists available at SciVerse ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Local analytic first integrals of planar analytic differential systems

Ilker E. Colak [a], Jaume Llibre [a], Claudia Valls [b,*]

[a] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
[b] Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, 1049-001 Lisboa, Portugal

ARTICLE INFO

Article history:
Received 7 October 2012
Received in revised form 10 January 2013
Accepted 1 March 2013
Available online 6 March 2013
Communicated by A.P. Fordy

Keywords:
Analytic first integral
Analytic differential system

ABSTRACT

We study the existence of local analytic first integrals of a class of analytic differential systems in the plane, obtained from the Chua's system studied in L.O. Chua (1992, 1995), N.V. Kuznetsov et al. (2011), G.A. Leonov et al. (2012) [6,7,11,13]. The method used can be applied to other analytic differential systems.

1. Introduction and statement of results

The nonlinear ordinary differential equations appear in many branches of applied mathematics, physics and, in general, in applied sciences. For a differential system defined on the plane $\mathbb{R}^2$ the existence of a first integral determines completely its phase portrait. Since for such vector fields the notion of integrability is based on the existence of a first integral the following natural question arises: *Given a differential system in $\mathbb{R}^2$, how to recognize if this system has a first integral*?

The easiest planar differential systems having a first integral are the Hamiltonian ones. The integrable planar differential systems which are not Hamiltonian are, in general, very difficult to detect. Many different methods have been used for studying the existence of first integrals for non-Hamiltonian differential systems based on: Noether symmetries [4], the Darbouxian theory of integrability [8], the Lie symmetries [16], the Painlevé analysis [3], the use of Lax pairs [12], the direct method [9] and [10], the linear compatibility analysis method [18], the Carlemann embedding procedure [5] and [2], the quasimonomial formalism [3], the Ziglin's method [19], the Morales–Ramis theory [15], etc.

The main objective of this Letter is to show how to study the existence or non-existence of analytic first integrals of planar analytic differential systems, when the standard theorems providing sufficient conditions for the non-existence do not work.

We consider analytic differential systems

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y), \tag{1}$$

defined in an open subset U of $\mathbb{R}^2$. We say that a non-constant analytic function $H : U \to \mathbb{R}^2$ is an *analytic first integral* of system (1) in U if H is constant on the solution curves of system (1), or equivalently

$$f(x, y)H_x + g(x, y)H_y = 0,$$

in U. Of course, H_x denotes the derivative of H with respect to x.

There exist well-known results providing sufficient conditions for the non-existence of local analytic first integrals of system (1), as for instance the following two theorems.

Theorem 1. *(See Poincaré [17].) Assume that the eigenvalues $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ at some singular point p of the analytic differential system (1) do not satisfy any resonance condition of the form*

$$\lambda_1 k_1 + \lambda_2 k_2 = 0,$$

for any positive integers k_1 and k_2. Then system (1) has no analytic first integrals defined in a neighborhood of p.

Theorem 2. *(See Li et al. [14].) Assume that the eigenvalues λ_1 and λ_2 at some singular point p of the analytic differential system (1) satisfy that $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Then system (1) has no analytic first integrals in a neighborhood of p if p is isolated in the set of all singular points of system (1).*

The problem for studying the non-existence of local analytic first integrals of system (1) in a neighborhood of a singular point appears when the sufficient conditions of Theorems 1 and 2 cannot be applied. In this work we deal with such a case. More precisely, we will study the local analytic integrability of the analytic differential system

* Corresponding author.
E-mail addresses: ilkercolak@mat.uab.cat (I.E. Colak), jllibre@mat.uab.cat (J. Llibre), cvalls@math.ist.utl.pt (C. Valls).

0375-9601/$ – see front matter
http://dx.doi.org/10.1016/j.physleta.2013.03.001

$$\dot{x} = y + m(\tanh x - x),$$
$$\dot{y} = -(\alpha + 1)y - \alpha m(\tanh x - x), \tag{2}$$

in a neighborhood of its unique singular point, the origin. Here α and m are real parameters. Our main result is the following.

Theorem 3. *The analytic differential system (2) has a local analytic first integral in a neighborhood of the origin if and only if $m = 0$. When $m = 0$, a first integral is $(\alpha + 1)x + y$.*

Theorem 3 is proved in Section 2.

In fact Theorem 2 can be partially applied to the differential system (2), because at the singular point located at the origin of coordinates the eigenvalues are 0 and $-1-\alpha$. Therefore, when $\alpha \neq -1$ and $m \neq 0$ (otherwise the origin is not isolated in the set of all singular points) Theorem 2 says that the differential system (2) has no local analytic first integral in a neighborhood of the origin. So, using Theorem 2, in order to complete the proof of Theorem 3 we need to show it when $\alpha = -1$. But since the proof of Theorem 3 for the case $\alpha = -1$ needs essentially the same computations that the proof of Theorem 3 for any value of α, we shall prove Theorem 3 in this last case.

The differential analytic system (2) comes from the Chua's system

$$\dot{X} = \alpha(Y - X) - \alpha\big(m_1 X + (m_0 - m_1)\tanh X\big),$$
$$\dot{Y} = X - Y + Z,$$
$$\dot{Z} = -(\beta Y + \gamma Z),$$

studied in [6,7,11,13], by doing the linear change of variables

$$x = X, \qquad y = \alpha(Y - X), \qquad z = 0,$$

and by defining $m = \alpha m_1$, when $m_0 = \beta = \gamma = 0$. In fact, in Chua's original system (see [6,7]) he works with a function which has a qualitative behavior as the function $\tanh x$, but in the related papers [11,13] the authors already work with the function $\tanh x$.

2. Proof of Theorem 3

If $m = 0$, it is easy to check that $(\alpha + 1)x + y$ is an analytic first integral of system (2).

The main idea of the proof is to assume that there exists a local analytical first integral H in a neighborhood of the origin of system (2) when $m \neq 0$. Then writing it in power series of the variables x and y, and forcing that it is a first integral we obtain a finite system of equations for each degree of the monomials. The unknowns of these infinitely many systems are the coefficients of the power series of H. Using induction we will show that all these coefficients vanish, and consequently such a first integral does not exist.

Assume $m \neq 0$ and suppose H is a local analytic first integral of system (2) in a neighborhood of the origin. Then we can write H as a Taylor series

$$H(x, y) = \sum_{i,j=0}^{\infty} a_{i,j} x^i y^j,$$

where $a_{0,0} = 0$. We also expand the hyperbolic tangent in its Taylor series as

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots$$
$$= \sum_{i=1}^{\infty} \frac{B_{2i} 4^i (4^i - 1)}{(2i)!} x^{2i-1} = \sum_{i=1}^{\infty} b_i x^{2i-1},$$

where the B_i are the Bernoulli numbers, see for instance [1]. We note that the numbers $b_i \neq 0$ for $i \geqslant 1$. Then, clearly

$$\tanh x - x = \sum_{i=2}^{\infty} b_i x^{2i-1}.$$

By definition H must satisfy the equation

$$\big(y + m(\tanh x - x)\big)H_x + \big(-(1+\alpha)y - \alpha m(\tanh x - x)\big)H_y = 0,$$

which can be rewritten as

$$G = m(\tanh x - x)(H_x - \alpha H_y) + y\big(H_x - (1+\alpha)H_y\big) = 0, \tag{3}$$

for all (x, y).

Using Eq. (3) we will prove by induction that

$$a_{i,n-i} = 0 \quad \text{for all } n \geqslant 1 \text{ and } i = 1, 2, \ldots, n, \tag{4}$$

which means that all the $a_{i,j}$ are zero, hence H is zero. We shall first compute the left-hand side of Eq. (3). We have

$$H_x = \sum_{i=1,j=0}^{\infty} i a_{i,j} x^{i-1} y^j = \sum_{i,j=0}^{\infty} (i+1) a_{i+1,j} x^i y^j,$$

$$H_y = \sum_{i=0,j=1}^{\infty} j a_{i,j} x^i y^{j-1} = \sum_{i,j=0}^{\infty} (j+1) a_{i,j+1} x^i y^j,$$

$$H_x - \alpha H_y = \sum_{i,j=0}^{\infty} \big((i+1)a_{i+1,j} - \alpha(j+1)a_{i,j+1}\big) x^i y^j$$
$$= \sum_{i,j=0}^{\infty} c_{i,j} x^i y^j, \tag{5}$$

$$H_x - (\alpha+1)H_y = \sum_{i,j=0}^{\infty} \big((i+1)a_{i+1,j} - (\alpha+1)(j+1)a_{i,j+1}\big) x^i y^j.$$

Then

$$(\tanh x - x)(H_x - \alpha H_y)$$
$$= \sum_{j=0}^{\infty} \big(b_2 c_{0,j} x^3 + b_2 c_{1,j} x^4 + (b_2 c_{2,j} + b_3 c_{0,j}) x^5 + (b_2 c_{3,j} + b_3 c_{1,j}) x^6 + \cdots\big) y^j$$
$$= \sum_{i=3,j=0}^{\infty} \left(\sum_{k=2}^{\lceil i/2 \rceil} b_k c_{i-2k+1,j} \right) x^i y^j, \tag{6}$$

where $\lceil x \rceil$ denotes the *ceiling function*, which gives the smallest integer greater than x. We also have

$$y\big(H_x - (\alpha+1)H_y\big)$$
$$= \sum_{i,j=0}^{\infty} \big((i+1)a_{i+1,j} - (\alpha+1)(j+1)a_{i,j+1}\big) x^i y^{j+1}$$
$$= \sum_{i=0,j=1}^{\infty} \big((i+1)a_{i+1,j-1} - (\alpha+1) j a_{i,j}\big) x^i y^j. \tag{7}$$

Using equalities (6) and (7), Eq. (3) becomes

$$G = \sum_{i=0,j=1}^{\infty} \big((i+1)a_{i+1,j-1} - (\alpha+1) j a_{i,j}\big) x^i y^j + m \sum_{i=3,j=0}^{\infty} \left(\sum_{k=2}^{\lceil i/2 \rceil} b_k c_{i-2k+1,j} \right) x^i y^j = 0.$$