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Information metric from Riemannian superspaces

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ABSTRACT

The Fisher information metric is introduced in order to find the meaning of the probability distribution in classical and quantum systems described by Riemannian non-degenerated superspaces. In particular, the physical rôle played by the coefficients $\mathbf{a}$ and $\mathbf{a}^*$ of the fermionic part of an emergent metric solution, obtained previously (Cirilo-Lombardo, 2012 [1]) is explored. We find that the metric solution of the superspace establishes a connexion between the Fisher metric and its quantum counterpart, corroborating early conjectures by Caianiello et al. This quantum mechanical extension of the Fisher metric is described by the CP^1 structure of the Fubini–Study metric, with coordinates $\mathbf{a}$ and $\mathbf{a}^*$.

1. Introduction

The problem of giving an unambiguous quantum mechanical description of a particle in a general spacetime has been repeatedly investigated. The introduction of supersymmetry provided a new approach to this question, however, some important aspects concerning the physical observables remain not completely understood, classically and quantically speaking.

The superspace concept, on the other hand, simplify considerably the link between ordinary relativistic systems and 'supersystems', extending the standard (bosonic) spacetime by means of a general (super)group manifold, equipped with also fermionic (odd) coordinates.

In a previous work [1] we introduced, besides other supersymmetric quantum systems of physical interest, a particular $N=1$ superspace [2]. That was made with the aim of studying a superworld-line quantum particle (analogously to the relativistic case) and its relation with SUGRA theories [5,2,3]. The main feature of this superspace is that the supermetric, which is the basic ingredient of a Volkov–Pashnev particle action [4,5], is *invertible and non-degenerate*, that is, of $G4$ type in the Casalbuoni classification [6]. As shown in [2,3], the non-degeneracy of the supermetrics (and therefore of the corresponding superspaces) leads to important consequences in the description of physical systems. In particular, notorious geometrical and topological effects on the quantum states, namely, *consistent mechanisms of localization and confinement*, due purely to the geometrical character of the Lagrangian. Also *an alternative to the Randall–Sundrum (RS) model without extra bosonic coordinates* can be consistently formulated in terms of such non-degenerated superspace approach, eliminating the problems that the RS-like models present at the quantum level [1,3]. Given the importance of the non-degeneracy of the supermetrics in the formulation of physical theories, in the present Letter we analyze further the super-line element introduced in [4,2], focusing on a probabilistic context. As the Lagrangian of this particular supermetric brings us localized states showing a Gaussian behaviour, it is of clear interest to analyze the probabilistic and information theoretical meaning of such a geometry. To this specific end, the Fisher metric [7] (Fisher–Rao in the quantum sense [8]) has been considered in several works in order to provide a geometrical interpretation of the statistical measures. Fisher's information measure (FIM) was advanced already in the 1920s decade, well before the advent of Information Theory (IT). Much interesting work has been devoted to the physical applications of FIM in recent times (see, for instance, [9] and references therein). In [12], a generalization of the Yang–Mills Hitchin proposal was made suggesting an indentification of the Lagrangian density with the Fisher probability distribution $(P(\theta))$. However, this idea was explored from a variational point of view, in previous work by Plastino et al. [10,11], and in several geometrical ways by Brody and Hughston in [17]. This proposal brought a contribution to the line of works looking for a connexion between the spacetime geometry and quantum field theories.

In the last decades it has been claimed that the above expectation is partially realized in the AdS/CFT (anti-de Sitter/Conformal Field Theory) correspondence [13], which asserts that the equivalence of a gravitational theory (i.e., the geometry of

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spacetime) and a conformal quantum field theory at the boundary of spacetime certainly exists.

The organization of this Letter is as follows: in Section 2, the Fisher metric and a new generalization of the Hitchin proposal are introduced. Section 3 presents global aspects of the $N = 1$ non-degenerate superspace solution of reference [1]. In Section 5 we analyze the 'bosonic' (B_0) part of our Fisher supermetric by 'turning off' all the fermions. Section 6 presents further discussions on our results, in connexion with a quantum extension of the Fisher metric. Finally, Section 7 is devoted to collecting the main results and our concluding remarks.

2. Fisher's metric and Hitchin's prescription

In general, the Fisher information metric[1] (more precisely, the Fisher–Rao metric) is a Riemannian metric for the manifold of the parameters of probability distributions. The Rao distance (geodesic distance in the parameter manifold) provides a measure of the difference between distinct distributions. In the thermodynamic context, the Fisher information metric is directly related to the rate of change in the corresponding order parameters and can be used as an information-geometric complexity measure for classifying phase transitions, e.g., the scalar curvature of the thermodynamic metric tensor diverges at (and only at) a phase transition point (this issue will be analyzed in future work). In particular, such relations identify second-order phase transitions via divergences of individual matrix elements.

The Fisher–Rao information metric is given by [7,8]

$$G_{ab}(\theta) = \int d^D x\, P(x;\theta)\partial_a \ln P(x;\theta)\partial_b \ln P(x;\theta) \tag{1}$$

where x^μ ($\mu, \nu = 0, \ldots, D$) are the random variables and θ_a ($a, b = 1, \ldots, N$) are the parameters of the probability distribution. Besides this, $P(x;\theta)$ must fulfill the normalization condition

$$\int d^D x\, P(x;\theta) = 1. \tag{2}$$

In his work [14], Hitchin proposed the use of the squared field strength of Yang–Mills theory as a probability distribution. A generalization of the Hitchin proposal [12] – see also [10] – consists in identifying the probability distribution with the *on-shell* Lagrangian density of a field theory

$$P(x;\theta) := -\mathcal{L}(x;\theta)|_{solution}. \tag{3}$$

In Ref. [12], they consider the case in which the distribution dependence on the variables takes the form $P(x,\theta) = P(x-\theta)$. Note that this is only possible in the cases in which the numbers of the spacetime dimensions x and the parameters θ coincide, that is, in which $D = N$.

In the present work we will consider two different approaches to the problem. First, we will follow the 'generalized Hitchin prescription', identifying our Lagrangian (calculated at the solution) with the probability distribution. Then, we will introduce a new proposal: we will take the state probability current of the emergent metric solution of the superspace (obtained explicitly in [1]) as being itself the probability distribution. The results of the two approaches will be compared in order to infer the physical meaning of the $\mathbf{a}$ and $\mathbf{a}^*$ parameters appearing in the pure fermionic part of the superspace metric.

[1] We prefer the term 'information metric' rather than 'emergent metric' because an 'emergent metric' appears as a quantum solution for the physical spacetime geometry while the Fisher information metric is related to statistical parameters and is, therefore, a probabilistic concept.

3. Emergent metric solution

The model introduced in [1] describes a free particle in a superspace with coordinates $z_A \equiv (x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. The corresponding Lagrangian density is

$$\mathcal{L} = -m\sqrt{\omega^A \omega_A} = -m\sqrt{\mathring{\omega}_\mu \mathring{\omega}^\mu + \mathbf{a}\dot{\theta}^\alpha \dot{\theta}_\alpha - \mathbf{a}^* \dot{\bar{\theta}}^{\dot{\alpha}} \dot{\bar{\theta}}_{\dot{\alpha}}}, \tag{4}$$

where $\mathring{\omega}_\mu = \dot{x}_\mu - i(\dot{\theta}\sigma_\mu\bar{\theta} - \theta\sigma_\mu\dot{\bar{\theta}})$, and the dot indicates derivative with respect to the evolution parameter τ, as usual. In coordinates, the line element of the superspace reads

$$\begin{aligned} ds^2 &= \dot{z}^A \dot{z}_A \\ &= \dot{x}^\mu \dot{x}_\mu - 2i\dot{x}^\mu(\dot{\theta}\sigma_\mu\bar{\theta} - \theta\sigma_\mu\dot{\bar{\theta}}) + (\mathbf{a} - \bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}})\dot{\theta}^\alpha\dot{\theta}_\alpha \\ &\quad - (\mathbf{a}^* + \theta^\alpha\theta_\alpha)\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}. \end{aligned} \tag{5}$$

The 'squared' solution with three compactified dimensions (λ spin fixed) is [1]

$$g_{AB}(t) = e^{A(t)+\xi\varrho(t)} g_{AB}(0), \tag{6}$$

where the initial values of the metric components are given by

$$g_{ab}(0) = \langle\psi(0)| \begin{pmatrix} a \\ a^\dagger \end{pmatrix}_{ab} |\psi(0)\rangle, \tag{7}$$

or, explicitly,

$$\begin{aligned} &g_{\mu\nu}(0) = \eta_{\mu\nu}, \qquad g_{\mu\alpha}(0) = -i\sigma_{\mu\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}, \\ &g_{\mu\dot{\alpha}}(0) = -i\theta^\alpha\sigma_{\mu\alpha\dot{\alpha}}, \qquad g_{\alpha\beta}(0) = (a - \bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}})\epsilon_{\alpha\beta}, \\ &g_{\dot{\alpha}\dot{\beta}}(0) = -(a^* + \theta^\alpha\theta_\alpha)\epsilon_{\dot{\alpha}\dot{\beta}}. \end{aligned} \tag{8}$$

It is worth mention here that these components were obtained in a simpler case in [5].

The bosonic and spinorial parts of the exponent in the superfield solution (6) are, respectively,

$$A(t) = -\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + c_1 t + c_2, \tag{9}$$

and

$$\begin{aligned} \xi\varrho(t) &= \xi(\phi_\alpha(t) + \bar{\chi}_{\dot{\alpha}}(t)) \\ &= \theta^\alpha\left(\mathring{\phi}_\alpha \cos(\omega t/2) + \frac{2}{\omega} Z_\alpha\right) \\ &\quad - \bar{\theta}^{\dot{\alpha}}\left(-\mathring{\bar{\phi}}_{\dot{\alpha}} \sin(\omega t/2) - \frac{2}{\omega}\bar{Z}_{\dot{\alpha}}\right) \\ &= \theta^\alpha\mathring{\phi}_\alpha \cos(\omega t/2) + \bar{\theta}^{\dot{\alpha}}\mathring{\bar{\phi}}_{\dot{\alpha}} \sin(\omega t/2) + 4|\mathbf{a}|\,\mathrm{Re}(\theta Z), \end{aligned} \tag{10}$$

where $\mathring{\phi}_\alpha$, Z_α, $\bar{Z}_{\dot{\beta}}$ are constant spinors, $\omega \approx 1/|\mathbf{a}|$ and the constant c_1, due to the obvious physical reasons and the chirality restoration of the superfield solution [1,2], should be taken purely imaginary.

4. Fisher's information metric from Riemannian's superspaces

The Fisher method considers a family of probability distributions, characterized by certain number of parameters. The metric components are then defined by considering derivatives in different 'directions' in the parameters space, that is, measuring 'how distant' two distinct sets of parameters put apart the corresponding probability distributions.

In the following we will calculate the Fisher information metric corresponding to a generalized Hitchin 'on-shell' Lagrangian prescription. In our case, the parameters of interest in the metric