Physics Letters A 375 (2011) 1179–1182

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

The dimension formula for the Lorenz attractor ☆

G.A. Leonov [a], A.Yu. Pogromsky [b,*], K.E. Starkov [c]

[a] Faculty of Mathematics and Mechanics, St. Petersburg State University, Universitetsky pr. 28, St. Petersburg 198504, Russia
[b] Department of Mechanical Engineering, Eindhoven University of Technology, POB 513, 5600MB, Eindhoven, The Netherlands
[c] CITEDI-IPN, Av. del Parque 1310, Mesa de Otay, Tijuana, 22510, BC, Mexico

ARTICLE INFO

Article history:
Received 25 October 2010
Received in revised form 12 January 2011
Accepted 16 January 2011
Available online 18 January 2011
Communicated by C.R. Doering

Keywords:
Lorenz system
Lyapunov dimension

ABSTRACT

An analytical formula for the Lyapunov dimension of the Lorenz attractor is presented under assumption that all the equilibria are unstable.

1. Introduction

Study of ergodic properties of nonlinear systems is one of important topics in nonlinear sciences. These properties can be partially characterized by various dimensions (fractal, Hausdorff, Lyapunov) of invariant compact sets. A lot of reports aimed at numerical evaluations of these dimensions for different systems has been published, while pure analytical estimates are more difficult to derive. With this in mind it is worth to acknowledge a pioneering work by Douady and Oesterlé [1], that developed technique for analytical estimation of various dimensions of invariant sets of dynamical systems. Later on a number of results was obtained in this direction, for a survey and references, see [2]. The famous Lorenz system serves as an benchmark example to test different techniques and the goal of this Letter is to show that a previously known result for the Lyapunov dimension holds true for a larger set of system parameters. In our study we apply a combination of the Douady–Oesterlé approach with the direct Lyapunov method. This combined technique was introduced in [3] and then developed in [4,7].

2. The Lyapunov dimension

Consider a continuously differentiable map F defined on an open set $U \subset \mathbb{R}^n$. Denote by T_xF its Jacobi matrix evaluated at point x. Assume that there is an invariant set $K \subset U$, $F(K) = K$.

Let $\alpha_i(A)$, $i = 1, \dots, n$, stand for singular values of a square matrix A ordered in the nonincreasing order. The local Lyapunov dimension (see [8]) of the map F at the point $x \in K$ is the number $\dim_L(F, x) = j + s$, where j is the largest integer from $[1, n]$ such that

$$\alpha_1(T_xF) \cdots \alpha_j(T_xF) \geqslant 1$$

and $s \in [0, 1)$ is such that

$$\alpha_1(T_xF) \cdots \alpha_j(T_xF)\alpha_{j+1}^s(T_xF) = 1.$$

The Lyapunov dimension of the map F on the set K is the number $\dim_L(F, K) = \sup_K \dim_L(F, x)$. The local Lyapunov dimension of the sequence of maps F^i at the point $x \in K$ is the number $\dim_L x = \limsup_{i\to\infty} \dim_L(F^i, x)$. The Lyapunov dimension of the sequence of maps F^i on K is the number $\dim_L K = \sup_K \dim_L x$. Similar definitions can be given for a one-parameter group of maps.

Consider the following system of differential equations

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{1}$$

with a continuously differentiable right-hand side that satisfies some regularity assumptions to ensure the existence of all solutions starting from $\mathbb{R}^n$ on the infinite time interval. Let F^t be the

☆ The research has received partial funding from the CONACYT project (Mexico) Analisis de localizacion de conjuntos compactos invariantes de systemas no lineales con dinamica compleja y sun aplicaciones, No. 000000000078890. The work was performed while K.S. visited Eindhoven University of Technology (TUE). He thanks TUE team for hospitality.

* Corresponding author.
E-mail addresses: leonov@math.spbu.ru (G.A. Leonov), a.pogromsky@tue.nl (A.Yu. Pogromsky), konst@citedi.mx (K.E. Starkov).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2011.01.034

shift map: $F^t(\xi) = x(t,\xi)$. Suppose that there is a bounded set K which is invariant with respect to the map F^t: $F^t(K) = K$. We denote by J the Jacobi matrix of the vector function f: $J(x) = \partial f(x)/\partial x$. For a given positive-definite symmetric matrix P. Denote by $\lambda_1(x) \geqslant \cdots \geqslant \lambda_n(x)$ solutions of the following algebraic equation

$$\det\big(PJ(x) + J^\top(x)P - \lambda P\big) = 0. \tag{2}$$

Finally, let $L_f v$ stand for the Lie derivative of the function v with respect to the vector field f: $L_f v(x) = \nabla v(x)^\top f(x)$. The following Theorem 1 is borrowed from [4], see also [10,11].

Theorem 1. *Suppose that for the integer $j \in [1,n]$ and $s \in [0,1)$ there is a positive-definite matrix P and a continuously differentiable on $\mathbb{R}^n$ function v such that*

$$\sup_K\big(\lambda_1(x_0) + \cdots + \lambda_j(x_0) + s\lambda_{j+1}(x_0) + L_f v(x_0)\big) < 0.$$

Then for sufficiently large $t > 0$ the inequality

$$\dim_L\big(F^t, K\big) < j + s$$

holds.

In essence, the proof of this theorem is based on ideas of Douady and Oesterlé [1]. The main difference consists in the presence of the term $L_f v(x_0)$. This term allows to replace integration along the system trajectories within the estimation of Lyapunov exponents by differentiation of some auxiliary function which is usually referred to as a Lyapunov function.

3. The main result

It is known that for some parameters r, σ, b of the Lorenz system

$$\dot{x} = -\sigma x + \sigma y,$$
$$\dot{y} = rx - y - xz,$$
$$\dot{z} = -bz + xy, \quad \sigma, b, r > 0. \tag{3}$$

The Lyapunov dimension of any invariant compact set is bounded from above by the local Lyapunov dimension of the origin (see [5,6] and [4,10,11], Theorem 10.4, Section 12.4 in [7], Section 5 in [9]). The goal of this Letter is prove the following statement, which is valid for the most interesting set of system parameters. Namely, we establish

Theorem 2. *Let K be an invariant compact set of the Lorenz system; $(0,0,0) \in K$. Assume that all the equilibria of (3) are hyperbolically unstable. Then*

$$\dim_L K = 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4r\sigma}}. \tag{4}$$

4. The proof of the main result

We formulate and prove a result which is more general than Theorem 2 and later on the main result is proven as a corollary.

Theorem 3. *Let K be an invariant compact set of the Lorenz system, $(0,0,0) \in K$. Assume that $r > 1$ and*

(i) $\mu := r\sigma + (b-1)(\sigma - b) > 0$.

(ii)

$$r\sigma^2\left(4 - \frac{b}{c}\right) + d + 2\sigma(b-1)(2\sigma - 3b) - b(b-1)^2 \geqslant 0, \tag{5}$$

where $c := \max\{1, b\}$ and $d := b\sigma^2(1-c)/c$. Then (4) holds.

Proof. The Jacobi matrix for the Lorenz system J is given by

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}. \tag{6}$$

As in [4], let us take

$$P = \begin{pmatrix} \frac{r\sigma + (b-1)(\sigma-1)}{\sigma^2} & -\frac{b-1}{\sigma} & 0 \\ -\frac{b-1}{\sigma} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

This matrix is positive definite if and only if $r\sigma + (b-1)(\sigma - b) > 0$. Denote

$$a := \sigma\big[r\sigma + (b-1)(\sigma - b)\big]^{-1/2}. \tag{8}$$

The generalized characteristic equation (2) for the given P takes the form

$$(\lambda + 2b) \times \left(\lambda^2 + 2(\sigma+1)\lambda - a^2\left(y + \frac{b-1}{\sigma}x\right)^2 + 4b(\sigma + 1 - b) - \left(\frac{2\sigma}{a} - az\right)^2\right) = 0,$$

with solutions $\lambda_{1,2,3}$ defined by

$$\lambda_{1,3} = -(\sigma+1) \pm \left((\sigma + 1 - 2b)^2 + \left(\frac{2\sigma}{a} - az\right)^2 + a^2\left(y + \frac{b-1}{\sigma}x\right)^2\right)^{\frac{1}{2}}, \quad \lambda_2 = -2b < 0.$$

Consider differences

$$\lambda_1 - \lambda_2 \geqslant -(\sigma + 1 - 2b) + \sqrt{(\sigma + 1 - 2b)^2} \geqslant 0,$$
$$\lambda_2 - \lambda_3 \geqslant (\sigma + 1 - 2b) + \sqrt{(\sigma + 1 - 2b)^2} \geqslant 0,$$

which prove inequalities $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$.

Notice that $\lambda_{1,3} = -(\sigma+1) \pm \phi$ where $\phi > 0$ and

$$\phi^2 = (\sigma - 1)^2 + 4\sigma r - 4\sigma z + a^2z^2 + a^2\left(y + \frac{b-1}{\sigma}x\right)^2.$$

Consider the relation

$$\beta := \lambda_1 + \lambda_2 + s\lambda_3 = -(\sigma + 1 + 2b) - s(\sigma + 1) + (1 - s)\phi(x, y, z).$$

Using the inequality

$$\sqrt{p + q} \leqslant \sqrt{p} + \frac{q}{2\sqrt{p}} \quad \forall p > 0,\ q \geqslant 0,$$

and positivity of $1 - s$ one can estimate

$$\beta \leqslant -(\sigma + 1 + 2b) - s(\sigma + 1) + (1 - s)\sqrt{(\sigma - 1)^2 + 4\sigma r} + 2\frac{(1 - s)w(x, y, z)}{\sqrt{(\sigma - 1)^2 + 4\sigma r}},$$

with $w(x, y, z)$ defined by

$$w(x, y, z) := -\sigma z + \frac{a^2z^2}{4} + \frac{a^2}{4}\left(\frac{b-1}{\sigma}x + y\right)^2.$$