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Bursting phenomena as well as the bifurcation mechanism in controlled Lorenz oscillator with two time scales

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ABSTRACT

A controlled Lorenz model with fast-slow effect has been established, in which there exist order gap between the variables associated with the controller and the original Lorenz oscillator, respectively. The conditions of fold bifurcation as well as Hopf bifurcation for the fast subsystem are derived to investigate the mechanism of the behaviors of the whole system. Two cases in which the equilibrium points of the fast subsystem behave in different characteristics have been considered, leading to different dynamical evolutions with the change of coupling strength. Several types of bursting phenomena, such as fold/fold burster, fold/Hopf burster, near-fold/Hopf burster, fold/near-Hopf buster have been observed. Theoretical analysis shows that the bifurcations points which connect the quiescent state and the repetitive spiking state agree well with the turning points of the trajectories of the bursters. Furthermore, the mechanism of the period-adding bifurcations, resulting in the rapid change of the period of the movements, is presented.

1. Introduction

As a typical simple chaotic model, the Lorenz system, representing three modes of the Oberbeck–Boussinesq equations for fluid convection in a two-dimensional layer heated from below, has attracted a lot of researchers to work on the dynamical evolution to chaos in the past decades [1–4]. It is found that three equilibrium points may exist in the system, the properties of which change with the variation of the parameters, while the trajectory of the system cycling around two foci may form a chaotic attractor as butterfly [5,6]. Obviously, from an implementation point of view, chaotic systems with simpler structures deserve more attention, for which 3D autonomous systems are of the lowest possible dimensions. Therefore, many 3D chaotic systems have been established, such as Chua's circuit [7,8], Chen system [9], etc. To stabilize the oscillations or to utilize chaos, how to control these systems to meet the need is an important task in nonlinear dynamics. For the Lorenz system, a lot of controlling schemes have been introduced, which may lead the dynamics settle down to the targets, such as an stable equilibrium point, a limit cycle or chaotic oscillation [10–12]. However, in most of the reports, the controllers are designed on the same time scale with the original systems. In this Letter, we consider the case when there exists order gap between the two time scales associated with the controller and the Lorenz oscillator, respectively.

Many dynamical systems in physics, chemistry, biology and geo-physics involve two time scales [13–16], which often behave in periodic state characterized by a combination of relatively large amplitude and nearly harmonic small amplitude oscillations, conventionally denoted by N^K with N and K corresponding to large and small amplitude oscillations, respectively. Generally, we say the system is in quiescent state (QS) stage when all the variables are at rest or exhibit small amplitude oscillations. The effect of two time scales may lead the systems to spiking state (SP), in which the variables may behave in large amplitude oscillations [17]. Bursting phenomena can be observed when the variables alternating between QS and SP. Two important bifurcations can be found associated with the bursters: bifurcation of a quiescent state that leads to repetitive spiking and bifurcation of a spiking attractor that leads to quiescence [18].

Here we design the controller, the variables of which change on much smaller time scale to investigate the dynamics of the whole system. Different types of bursters as well as the mechanism will be presented and some new phenomena, such as sequence of period-adding bifurcations will be explained in details.

2. Mathematical model of controlled Lorenz oscillator

For the Lorenz oscillator, we can design a controller such that the full system can be written in the form

$$\dot{x} = \sigma(y - x) + \alpha u, \qquad \dot{y} = \rho x - y - xz - \alpha v,$$
$$\dot{z} = -\beta z + xy,$$

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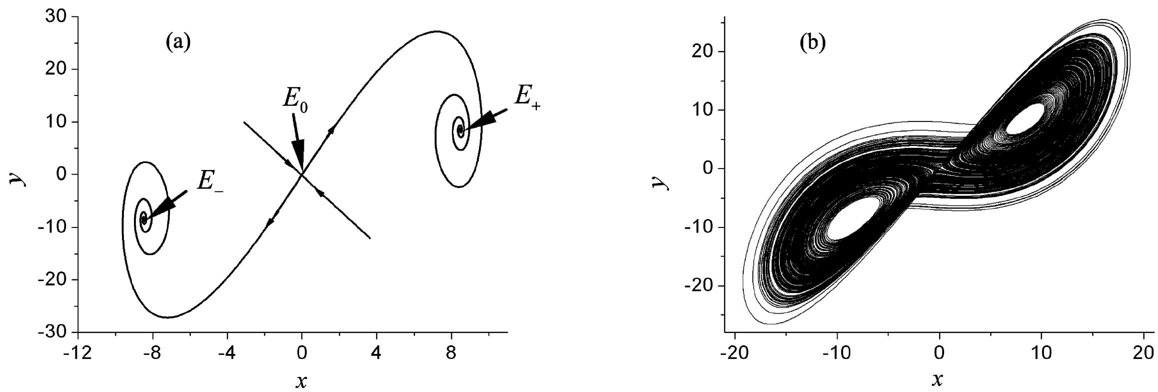

Fig. 1. Phase portraits for (a) $\sigma = 1$; (b) $\sigma = 10$.

$$\dot{u} = \varepsilon(yv - xz), \qquad \dot{v} = \varepsilon(x - y)u, \tag{1}$$

where ε is introduced to describe the order difference between the two time scales related to the original state variables and the controlling variables with the controlling strength α.

Obviously, when $\alpha = 0$, the full system reduces to Lorenz oscillator. The zero point, denoted by $E_0(0, 0, 0)$, is always the equilibrium point of the vector field associated with the characteristic equation, written in the form

$$(\lambda + \beta)\left[\lambda^2 + (\sigma + 1)\lambda - \sigma(\rho - 1)\right] = 0, \tag{2}$$

which suggests that E_0 is stable only for $\beta > 0$, $\sigma + 1 > 0$ and $-\sigma(\rho - 1) > 0$. However, for $\beta(\rho - 1) > 0$, other two equilibrium points can be observed, denoted by $E_\pm$ $[\pm\sqrt{\beta(\rho - 1)}, \pm\sqrt{\beta(\rho - 1)}, (\rho - 1)]$, the stabilities of which can be determined by

$$\lambda^3 + (\sigma + \beta + 1)\lambda^2 + \beta(\sigma + \rho)\lambda + 2\sigma\beta(\rho - 1) = 0, \tag{3}$$

implying that both $E_\pm$ are stable when the parameters satisfy $\sigma + \beta + 1 > 0$, $2\sigma\beta(\rho - 1) > 0$ and $\sigma^2 + (\beta - \rho + 3)\sigma + \rho(\beta + 1) > 0$.

For the parameters fixed at $\beta = 8/3$, $\rho = 28$ in this Letter, when $\alpha = 0$, all the three equilibrium points, i.e., E_0 and $E_\pm$, can be observed, the characteristics of which can be further determined by the parameter σ. Here we consider two cases with $\sigma = 1$ and $\sigma = 10$. For $\sigma = 1$, one may find that E_0 is a saddle point, while both $E_\pm$ are stable foci with the eigenvalues approximated at $(-2, -1.33 \pm 8.38I)$ (see Fig. 1(a)). The phase space is divided into two regions, corresponding to the attracting basins of the two foci, respectively. While for $\sigma = 10$, all the equilibrium points become unstable, and the trajectory forms an chaotic attractor (see Fig. 1(b)).

However, when $\alpha \neq 0$, the dynamics associated with two time scales may interact with each other to exhibit rich nonlinear behaviors, especially, various types of bursters, which will be described in the following.

3. Bifurcation analysis of the fast subsystem

We now turn to the investigation of the influence of the two time scales on the dynamical behaviors of the controlled Lorenz oscillator. Here we take $\varepsilon = 0.05$ to display the order gap between the two non-dimensional time scales and the full system can then be considered as the coupling of two subsystems, i.e., the fast subsystem (FS) and slow subsystem (SS) with coupling strength α.

For the fast subsystem, the equilibrium points, denoted by $E_q(x_0, y_0, z_0)$, can be determined by $F_0 = 0$, with

$$F_0 = \frac{3}{8}x_0^3 - \frac{3}{8\sigma}\alpha u x_0^2 - 27x_0 - \frac{1}{\sigma}\alpha u + \alpha v, \tag{4}$$

and $y_0 = x_0 - \frac{1}{\sigma}\alpha u$, $z_0 = \frac{3}{8\sigma}x_0(\sigma x_0 - \alpha u)$. Differentiating F_0 with respect to x_0 yields

$$F_1 = \frac{9}{8}x_0^2 - \frac{3\alpha u x_0}{4\sigma} - 27, \tag{5}$$

which gives two zero points, written as

$$x_a = \frac{1}{3\sigma}\left(\alpha u + \sqrt{\alpha^2 u^2 + 216\sigma^2}\right),$$
$$x_b = \frac{1}{3\sigma}\left(\alpha u - \sqrt{\alpha^2 u^2 + 216\sigma^2}\right), \tag{6}$$

at which the function F_0 reaches the extreme values, expressed by

$$F_a = -\frac{\alpha^3 u^3}{36\sigma^3} - \frac{10\alpha u}{\sigma} + \alpha v - \frac{1}{36\sigma^3}\left(\alpha^2 u^2 + 216\sigma^2\right)\sqrt{\alpha^2 u^2 + 216\sigma^2},$$
$$F_b = -\frac{\alpha^3 u^3}{36\sigma^3} - \frac{10\alpha u}{\sigma} + \alpha v + \frac{1}{36\sigma^3}\left(\alpha^2 u^2 + 216\sigma^2\right)\sqrt{\alpha^2 u^2 + 216\sigma^2}, \tag{7}$$

respectively. It is easy to check that $F_b > F_a$, and for $F_b < 0$ or $F_a > 0$, only one equilibrium point can be observed, while for $F_a < 0 < F_b$, three equilibrium points exist. Critical phenomenon occurs for $F_a = 0$ or $F_b = 0$, at which two equilibrium points meet with each other to form a degenerate equilibrium point corresponding to fold bifurcation.

The stability of E_q can be determined by the associated characteristic equation, written in the form

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{8}$$

where $a_2 = \frac{11}{3} + \sigma$, $a_1 = \frac{8}{3} - \frac{73\sigma}{3} + (1 + \frac{3\sigma}{8})x_0^2 - \frac{3}{8}\alpha u x_0$, $a_0 = -72\sigma + 3\sigma x_0^2 - 2\alpha u x_0$, implying that E_q is stable only for $a_0 > 0$, $a_2 > 0$ and $a_1 a_2 - a_0 > 0$. Instability of E_q may cause the bifurcations of the fast subsystem.

In the following, we only give the bifurcation results for $\sigma = 1$, while the bifurcation sets for $\sigma = 10$ can be derived accordingly.

3.1. Fold bifurcation

Note that the equilibrium points of the fast subsystem may change from three to one with the variation of the parameters. The critical condition corresponds to $F_a = 0$ or $F_b = 0$. Small perturbation of the parameters may cause the degenerate equilibrium point to disappear or to split into two different types of equilibrium points, implying fold bifurcation occurs. Therefore, the critical condition for the fold bifurcation can be expressed in the form