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Structural stochastic multiresonance in a hierarchical network of coupled threshold elements

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ABSTRACT

Structural stochastic multiresonance is observed in hierarchical networks with a tree-like structure, with non-dynamical threshold elements placed in the nodes and interacting along the edges of the network, driven by a common slowly varying periodic signal and independent noise sources. If the amplitude of the periodic signal is close to the threshold double maxima of the signal-to-noise ratio in the output of the top elements occur for a narrow range of the strength of connections, depending on the number of layers and the branching factor of the network. Additional maximum occurs at small noise intensities due to averaging of inputs received by the top elements from many branches of the network.

1. Introduction

Stochastic resonance (SR) occurs in certain nonlinear systems driven by a periodic signal and noise and is characterized by the enhancement of a periodic component of an output signal for non-zero noise intensity [1] (for review see [2,3]). In particular the output signal-to-noise ratio (SNR), defined as the ratio of the height of the peak in the output power spectrum density at the frequency of the input signal to the height of the noise background, shows maximum as a function of the noise intensity. In Refs. [4,5] a phenomenon called stochastic multiresonance (SMR) was described, characterized by the occurrence of more than one maximum of the SNR. Various kinds of SMR have been observed in low-dimensional systems, e.g., in potential systems with certain symmetry properties [4,5], chaotic systems with fractal basins of attraction [6], monostable overdamped systems [7], short chains of coupled FitzHugh–Nagumo oscillators [8] and the harmonic oscillator with fluctuating frequency [9]. In Ref. [10] SMR was observed also in a complex system, the Ising model on a scale-free network driven by a weak periodic magnetic field, with thermal fluctuations playing the role of noise and the time-dependent magnetization assumed as the output signal. In the latter case the origin of the multiresonance could be traced back to the structure of the exchange interactions, thus this phenomenon was called structural SMR.

In this Letter it is shown that structural SMR occurs in a much simpler system consisting of non-dynamical threshold elements (artificial neurons) driven by a common periodic signal and independent noise sources, placed in the nodes of a hierarchical network with a tree-like structure and connected (in a way typical of artificial neural networks) along its edges. It is known that SR can be observed in a single threshold element driven by a subthreshold periodic signal and noise [11]. In contrast, the elements in the upper layers of the hierarchical network exhibit SMR (double maxima of the SNR) in a narrow range of parameters such as the amplitude of the periodic signal and strength of connections, depending on the number of layers and the branching factor of the network. Hierarchical networks are ubiquitous in many branches of science as biology (neural networks), economy (networks of bank bankruptcies), communications (the Internet), [12], etc., thus studying properties of SR may extend our understanding of the ways of signal detection and transmission in such complex structures.

2. The model and methods of analysis

Arrays of coupled non-dynamical threshold elements are often used to investigate properties of SR in complex systems [13–15]. In this Letter the model under study is a system of threshold elements, with the threshold b, placed in the nodes of a hierarchical network with a tree-like structure which consists of $r+1$ layers, with l-th layer $l = 0, 1, 2, \dots, r$ containing n^{r-l} nodes, where n is the branching factor (Fig. 1). The edges of the network are directed, so that groups of n nodes in the layer $l-1$ are connected

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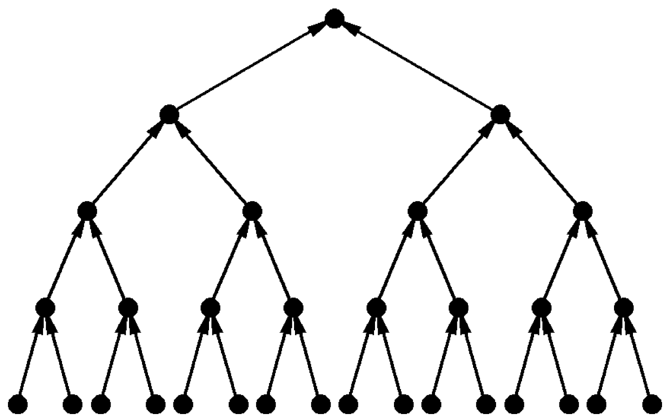

Fig. 1. A hierarchical network with a tree-like structure, with $r = 4$ and $n = 2$.

to one node in the layer l, and correspond to non-zero unidirectional interactions between the threshold elements. The form of the interactions is typical of neural networks, with the connection strength w. The elements are driven by a common periodic signal with amplitude A and frequency ω_s and independent Gaussian noises with intensity D. The output $y_l^{(i)}$ of the element in the node i belonging to the layer l at a discrete time step t is given by

$$y_0^{(i)}(t) = \Theta\big[A\sin\omega_s t + D\eta_0^{(i)}(t) - b\big], \quad i = 1, 2, \ldots, n^r; \tag{1}$$

$$y_l^{(i)}(t) = \Theta\Bigg[A\sin\omega_s t + D\eta_l^{(i)}(t) + \frac{w}{n}\sum_{j=1}^{n} y_{l-1}^{(n(i-1)+j)}(t-1) - b\Bigg],$$
$$l = 1, 2, \ldots, r, \quad i = 1, 2, \ldots, n^{r-l}, \tag{2}$$

where $\Theta(\cdot)$ is the Heaviside unit step function and $\langle\eta_k^{(i)}\eta_l^{(j)}\rangle = \delta_{k,l}\delta_{i,j}$.

The SNR in the element i belonging to the layer l can be evaluated both numerically, from the time series from Eqs. (1), (2), and analytically. The probability $P_l(t)$ that $y_l^{(i)}(t) = 1$ for any element i belonging to the lowest layer $l = 0$ is

$$P_0(t) = \Pr\big[A\sin\omega_s t + D\eta(t) - b > 0\big] = \frac{1}{2}\,\mathrm{erfc}\bigg[\frac{1}{D\sqrt{2}}(b - A\sin\omega_s t)\bigg], \tag{3}$$

and to the layer $l = 1, 2, \ldots, r$

$$P_l(t) = \sum_{k=0}^{n}\binom{n}{k}\Pi_k(t)P_{l-1}^k(t-1)\big[1 - P_{l-1}(t-1)\big]^{n-k}, \tag{4}$$

where $\Pi_k(t)$ is the conditional probability that $y_l^{(i)}(t) = 1$ provided that k of n elements coupled to the element under study are excited,

$$\Pi_k(t) = \Pr\bigg[A\sin\omega_s t + D\eta(t) + w\frac{k}{n} - b > 0\bigg] = \frac{1}{2}\,\mathrm{erfc}\bigg[\frac{1}{D\sqrt{2}}\bigg(b - A\sin\omega_s t - w\frac{k}{n}\bigg)\bigg] \tag{5}$$

(in Eqs. (3), (5) $\eta(t)$ denotes white Gaussian noise with unit variance). The output SNR in the element belonging to the layer l can be then evaluated as [11]

$$\mathrm{SNR} = \frac{|\hat{P}_l(\omega_s)|^2}{\langle P_l(t)\rangle - \langle P_l^2(t)\rangle}, \tag{6}$$

where $\hat{P}_l(\omega) = T_s^{-1}\sum_{t=0}^{T_s-1} P_l(t)\exp(i\omega t)$, with $T_s = 2\pi/\omega_s$, denotes the Fourier transform of $P_l(t)$, and the brackets denote the time average.

3. Results and discussion

Exemplary curves SNR vs. D obtained from the output signal of the threshold element in the top (that with $l = r$) layer of the network for fixed w, b, n, small frequency of the periodic signal and several values of r are shown in Fig. 2 for increasing amplitude of the periodic signal A. Theoretical curves were obtained from Eq. (4) in the adiabatic approximation $P_l(t-1) \approx P_l(t)$ valid for $\omega_s \to 0$. They are in agreement with numerical results apart from the region $D \to 0$ where the statistics for evaluating SNR from the simulated time series is poor. SR is observed for subthreshold periodic signals with $A < b$, i.e., the SNR has at least one maximum as a function of the noise intensity (Fig. 2(a, b)). Usually there is only one maximum of the SNR, located close to that for a single uncoupled element (Fig. 2(a)) and slightly higher, which is a typical signature of array-enhanced SR [16]. However, for a narrow range of A just below the threshold, depending on the connection strength w, additional maximum of the SNR appears at small D, which eventually becomes dominant as r is increased, i.e., SMR occurs (Fig. 2(b)). SR disappears for suprathreshold periodic signals with $A > b$ since the SNR diverges to infinity for decreasing noise intensity and the curves SNR vs. D may exhibit only local maxima at $D > 0$. Such "residual" SR is known to appear both in dynamical bistable systems, where its occurrence is ascribed to a resonant trapping mechanism [17] and in non-dynamical threshold systems like the Schmitt trigger [18] which do not provide any trapping condition by themselves. Anyway, for A just above the threshold and $r > 0$ two local maxima of the SNR may occur (Fig. 2(c)). As the amplitude of the periodic signal is further increased the local maxima eventually disappear and the SNR becomes a monotonically decreasing function of the noise intensity (Fig. 2(d)).

From Fig. 3(a) it can be seen that for fixed ω_s, b, n double maxima of the SNR in the top element occur if the amplitude of the periodic signal A and the connection strength w are within a triangle-like region whose size increases with r. Similarly, for fixed r the size of this region increases with the branching factor n (Fig. 3(b–d)). SMR is observed for $A < b$, while for $A > b$ the two maxima of the SNR are only local ones since the SNR diverges for $D \to 0$. In particular, SMR may occur in one-dimensional chains of threshold elements, i.e., in networks with $n = 1$ (Fig. 3(b–d)), even if the chain consists of only two elements ($r = 1$, Fig. 3(b)).

The appearance of the additional maximum of the SNR at small noise intensities is obviously related to the structure of the network and thus the observed phenomenon is an example of structural SMR. The origin of the above-mentioned maximum can be explained as follows. It was verified that this maximum diminishes and eventually disappears as ω_s is increased, thus SMR occurs only in the adiabatic limit of slowly varying periodic signals. Moreover, for any r the necessary (but not sufficient) condition for the occurrence of this maximum is $A + w/n > b$ (Fig. 3(a)). If the amplitude A is very close to the threshold b then even for small D there is a non-negligible probability that at least one element in the network is excited when the periodic signal reaches its maximum value, in particular if r, and thus the number of elements in the network, is large enough. The impulse from this element is then transmitted to the element in the upper layer which also with high probability becomes excited, since the sum of the slowly varying periodic signal, which is still close to maximum, and the input from the elements in the lower layer exceeds the threshold; and the noise is weak, thus the probability that it assumes a large negative value and prevents the element from being excited is negligible. Thus, provided that $r \ll T_s$, the output signal from any excited element