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Kochen–Specker theorem for a three-qubit system: A state-dependent proof with seventeen rays

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ABSTRACT

We consider Kochen–Specker theorem for three-qubit system with eight-dimensional state space. Reexamining the proof given by Kernaghan and Peres, we make some clarifications on the orthogonality of rays and rank-two projectors found by them. Basing on their five groups of orthogonal octad, we then show a proof that requires only seventeen rays.

1. Introduction

One of the main theorems on the (im)possibility of hidden variables in quantum mechanics is Kochen–Specker (KS) theorem. This theorem states that any hidden variable theory that agrees with quantum mechanics must be contextual. More specifically, it asserts that in the Hilbert space of dimension greater than two, it is impossible to associate definite numerical values, 1 or 0, with every projection operator $\hat{P}_m$ in such a way that if a set of commuting $\hat{P}_m$ satisfies $\sum_m \hat{P}_m = \hat{1}$, the corresponding values $v(\hat{P}_m)$ associated to their measurement results, will also satisfy $\sum_m v(\hat{P}_m) = 1$.

Since the first proof of Kochen and Specker [1] that uses 117 vectors in $\mathbb{R}^3$, there were many attempts to reduce the number of vectors either by conceiving ingenious models or extending the system considered to be of higher dimensions. It is known that there are different types of proofs e.g. state-independent proof and state-specific proof among others, such as discussed in the paper of [2], which gives the record-breaking proofs for a four-dimensional state space. Given the current progress in quantum information for finite-dimensional state spaces, there are also renewed interests in the KS theorem. Most notably are the recent beautiful experiments that have been conducted demonstrating KS-based state-independent quantum contextuality [3,4].

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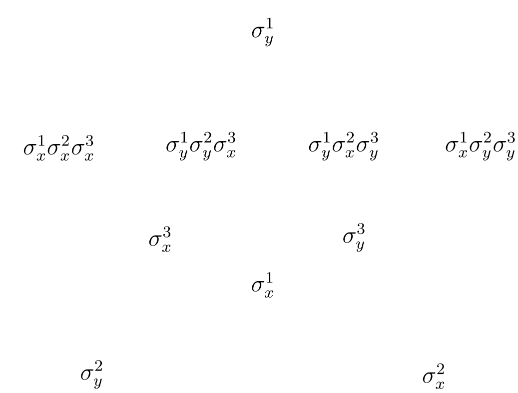

Fig. 1. Mermin's pentagram of commuting operators.

In this Letter, we will use the well-known fact that Pauli matrices are related to spin-half systems and hence generally to qubits (alluded earlier to be of interest in quantum information) and then sketch various proofs for three-qubit system in an eight-dimensional state space and then show our simple proof using only 17 and 18 rays.

2. KS-proof based on product rule

Mermin has proposed [5] for a three-qubit system, a set of ten mutually commuting operators that can be nicely arranged into lines of a pentagram as shown in Fig. 1.

With these operators, he provided a very simple KS theorem proof based on the five measurement contexts shown by the lines of the pentagram. The product of the four operators on each line gives an eight-dimensional identity matrix I except for the horizontal line which gives $-I$ i.e.

$$\begin{aligned}
&(\sigma_y^1)(\sigma_y^2)(\sigma_x^3)(\sigma_y^1\sigma_y^2\sigma_x^3) = I,\\
&(\sigma_y^1)(\sigma_x^2)(\sigma_y^3)(\sigma_y^1\sigma_x^2\sigma_y^3) = I,\\
&(\sigma_x^1)(\sigma_y^2)(\sigma_y^3)(\sigma_x^1\sigma_y^2\sigma_y^3) = I,\\
&(\sigma_x^1)(\sigma_x^2)(\sigma_x^3)(\sigma_x^1\sigma_x^2\sigma_x^3) = I,\\
&(\sigma_y^1\sigma_y^2\sigma_x^3)(\sigma_y^1\sigma_x^2\sigma_y^3)(\sigma_x^1\sigma_y^2\sigma_y^3)(\sigma_x^1\sigma_x^2\sigma_x^3) = -I, \qquad (1)
\end{aligned}$$

where σ_j^i is the j-th Pauli matrix tensored in the ith position or corresponding equivalently to the ith qubit.

According to product rule [6], the following equations for value functions should be satisfied:

$$\begin{aligned}
&v(\sigma_y^1)v(\sigma_y^2)v(\sigma_x^3)v(\sigma_y^1\sigma_y^2\sigma_x^3) = v(I),\\
&v(\sigma_y^1)v(\sigma_x^2)v(\sigma_y^3)v(\sigma_y^1\sigma_x^2\sigma_y^3) = v(I),\\
&v(\sigma_x^1)v(\sigma_y^2)v(\sigma_y^3)v(\sigma_x^1\sigma_y^2\sigma_y^3) = v(I),\\
&v(\sigma_x^1)v(\sigma_x^2)v(\sigma_x^3)v(\sigma_x^1\sigma_x^2\sigma_x^3) = v(I),\\
&v(\sigma_y^1\sigma_y^2\sigma_x^3)v(\sigma_y^1\sigma_x^2\sigma_y^3)v(\sigma_x^1\sigma_y^2\sigma_y^3)v(\sigma_x^1\sigma_x^2\sigma_x^3) = v(-I). \qquad (2)
\end{aligned}$$

The possible values assigned by value function $v(\cdot)$ is either 1 or -1, signifying a 'yes' or 'no' answer to the proposition tested via the measurement process. We assume the value returned from the measurement is determined by the hidden variables as opposed to its creation while the measurement is carried out. We further assumed that the value returned from the measurement does not depend on the measurement context. This means that if for instance $v(\sigma_y^1) = 1$ in the first equation of (2), it must also carry the value 1 in the second equation of (2). Under these two assumptions, Eqs. (2) cannot be consistently satisfied without a contradiction. As $v(I) = 1$ and $v(-I) = -1$, the sum of the right-hand side is an odd number and to get an odd number sum of the left-hand side, some of the equalities must be violated unless some of the value functions are allowed to map to different values in a different measurement context. It is possible that the above contradiction does not exist if the values are indeed created during the measurements. Since we are considering the possibility of hidden variables, the above contradiction means that the hidden variable theory that agrees with quantum mechanics must be contextual.

3. KS-proof based on sum rule

Adopting a method by Peres [7], we can take the logarithm of the functions of operators on the left-hand side of Eq. (1) and give a proof based on the sum rule [8] for the three-qubit system. Taking the logarithm of an operator $\hat{\Omega}$ and divide it by $i\pi$, tantamount to a change in the operator to $\frac{1}{2}(I - \hat{\Omega})$, which is again an operator. On the other hand, by using spectral decomposition, $F(\hat{\Omega}) = \sum_{i=1}^{d} F(\lambda_i)\hat{P}_i$ with $d = 8$ and using $\ln(-1) = i\pi$, $\ln(1) = 0$, it can be shown that

$$\frac{1}{i\pi}\ln(\hat{\Omega}) = \hat{P}_1 + \hat{P}_2 + \hat{P}_3 + \hat{P}_4, \qquad (3)$$

where $\hat{P}_1$, $\hat{P}_2$, $\hat{P}_3$ and $\hat{P}_4$ are projection operators of $\hat{\Omega}$ associated to eigenvalues -1. Hence, there are three ways of constructing the sum of operators for each line in the pentagram. Consider now two of them, namely

$$\begin{aligned}
&(i\pi)^{-1}\big(\ln(\sigma_y^2) + \ln(\sigma_x^3) + \ln(\sigma_y^1\sigma_y^2\sigma_x^3) + \ln(\sigma_y^1)\big),\\
&(i\pi)^{-1}\big(\ln(\sigma_y^1) + \ln(\sigma_y^1\sigma_x^2\sigma_y^3) + \ln(\sigma_y^3) + \ln(\sigma_x^2)\big),\\
&(i\pi)^{-1}\big(\ln(\sigma_x^2) + \ln(\sigma_x^1) + \ln(\sigma_x^3) + \ln(\sigma_x^1\sigma_x^2\sigma_x^3)\big),\\
&(i\pi)^{-1}\big(\ln(\sigma_x^1\sigma_x^2\sigma_x^3) + \ln(\sigma_y^1\sigma_y^2\sigma_x^3) + \ln(\sigma_y^1\sigma_x^2\sigma_y^3)\\
&\quad + \ln(\sigma_x^1\sigma_y^2\sigma_y^3)\big),\\
&(i\pi)^{-1}\big(\ln(\sigma_x^1\sigma_y^2\sigma_y^3) + \ln(\sigma_y^3) + \ln(\sigma_x^1) + \ln(\sigma_y^2)\big), \qquad (4)
\end{aligned}$$

and

$$\begin{aligned}
&\hat{P}_{\sigma_y^2} + \hat{P}_{\sigma_x^3} + \hat{P}_{\sigma_y^1\sigma_y^2\sigma_x^3} + \hat{P}_{\sigma_y^1},\\
&\hat{P}_{\sigma_y^1} + \hat{P}_{\sigma_y^1\sigma_x^2\sigma_y^3} + \hat{P}_{\sigma_y^3} + \hat{P}_{\sigma_x^2},\\
&\hat{P}_{\sigma_x^2} + \hat{P}_{\sigma_x^1} + \hat{P}_{\sigma_x^3} + \hat{P}_{\sigma_x^1\sigma_x^2\sigma_x^3},\\
&\hat{P}_{\sigma_x^1\sigma_x^2\sigma_x^3} + \hat{P}_{\sigma_y^1\sigma_y^2\sigma_x^3} + \hat{P}_{\sigma_y^1\sigma_x^2\sigma_y^3} + \hat{P}_{\sigma_x^1\sigma_y^2\sigma_y^3},\\
&\hat{P}_{\sigma_x^1\sigma_x^2\sigma_y^3} + \hat{P}_{\sigma_y^3} + \hat{P}_{\sigma_x^1} + \hat{P}_{\sigma_y^2}. \qquad (5)
\end{aligned}$$

Each expression in (4) gives rise to a matrix whose eigenvalues are 0, 2, and 4, except the fourth line that gives eigenvalues 1 and 3. Therefore the sum of their eigenvalues must be an odd number. The sum of eigenvalues in (5) also gives an odd number because (4) and (5) are constructed from the same measurement context. However, as each projection operator in (5) takes values 0 or 1 and they are repeated twice in different measurement contexts, the summation of values in (5) will result in an even number. Hence, the required contradiction.

4. Proof of Kernaghan and Peres I: octads and orthogonal rays

In the proof of Kernaghan and Peres [9], they considered four mutually commuting operators given below:

$$\begin{aligned}
&A = \sigma_{1z} \otimes \sigma_{2z} \otimes \sigma_{3z},\\
&B = \sigma_{1z} \otimes \sigma_{2x} \otimes \sigma_{3x},\\
&C = \sigma_{1x} \otimes \sigma_{2z} \otimes \sigma_{3x},\\
&D = \sigma_{1x} \otimes \sigma_{2x} \otimes \sigma_{3z}, \qquad (6)
\end{aligned}$$

and the product of these four operators is

$$ABCD = -1. \qquad (7)$$

These five equations in (6) and (7) generate five groups of orthogonal octads and they are labelled as R1, ..., R40 in Table 1. The coordinates of these rays are the same as given by Kernaghan and Peres [9] and can be found in their Table 1. The eleven octads proposed by them for their proof, labeled them as O1, ..., O11, are shown in the Table 1 below with rays belonging to each octad indicated. We found some non-orthogonal rays in the original table (can be easily checked) and are replaced here with orthogonal ones. Those rays in our Table 1, which are different from the original are ticked instead of crossed.

Taking the out product of the rays in Table 1, will give matrices of projection operators. For each octad, summing the eight projection operators will give eight-dimensional identity (listed from left to right according to Table 1) i.e.

$$\begin{aligned}
&\hat{P}_2 + \hat{P}_3 + \hat{P}_5 + \hat{P}_8 + \hat{P}_{33} + \hat{P}_{34} + \hat{P}_{35} + \hat{P}_{36} = \hat{I},\\
&\vdots\\
&\hat{P}_{18} + \hat{P}_{20} + \hat{P}_{22} + \hat{P}_{24} + \hat{P}_{25} + \hat{P}_{27} + \hat{P}_{29} + \hat{P}_{31} = \hat{I}, \qquad (8)
\end{aligned}$$

where $\hat{P}_i$ is the projection operator constructed from the ray R_i. According to the sum rule, we should have