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# Phase shift in the Whitham zone for the Gurevich–Pitaevskii special solution of the Korteweg–de Vries equation

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ABSTRACT

We get the leading term of the Gurevich–Pitaevskii special solution of the KdV equation in the oscillation zone without using averaging methods.



## 1. Introduction

The Gurevich–Pitaevskii (GP) special universal solution of the Korteweg–de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1.1}$$

was introduced in [1] in connection with the problem of description of collisionless shock waves (Sagdeev showed in [2] that such waves are of oscillating character). The behaviour of the GP special solution for $t \to -\infty$ and $x \to \pm\infty$ is determined in the main from the cubic canonical equation of the cusp catastrophe

$$x - tu + u^3 = 0. \tag{1.2}$$

The GP solution to the KdV equation is one of the most interesting special functions of the modern nonlinear mathematical physics.

In [1] it is shown that in problems of dispersion hydrodynamics (in particular, in problems of plasma theory) the GP special solution appears near the points of overturning of simple waves. From the results of [3–5] one actually sees that the same universal special function appears near the points of overturning of the generic state solutions to diverse dispersion perturbations of the equations of one-dimensional motion of ideal incompressible liquid

$$\rho'_t + (v\rho)'_x = 0,$$
$$v'_t + vv'_x + \alpha(\rho)\rho'_x = 0.$$

Here, $\rho$ is the density of the liquid, $v$ the velocity and $\alpha(\rho) = (c(\rho))^2/\rho$, where $c(\rho) = \sqrt{p'(\rho)}$ is the speed of sound and $p(\rho)$ the pressure. In particular, this is the case for solutions of the shallow water equations

$$h'_t + \left(hA'_x\right)'_x = \varepsilon^2\left(h^3A''_{xx}\right)''_{xx} + O\left(\varepsilon^4\right),$$
$$A'_t + \frac{1}{2}\left(A'_x\right)^2 + gh = \frac{1}{2}\varepsilon^2\left(A'''_{xxt} + A'_xA'''_{xxx} - \left(A''_{xx}\right)^2\right) + O\left(\varepsilon^4\right),$$

where $h$ is the free boundary, $A$ the potential of bottom velocity and $g$ the acceleration of gravity. The right-hand sides can actually be written as complete series in powers of the parameter $\varepsilon$ by the procedure given for instance in [6, Ch. 1, §4] (and not only as the so-called second approximations, as stated in [4]).

In the 1990s there were discovered surprising connections of the GP special solutions with some problems of quantum gravity. In [7] this solution was showed to simultaneously satisfy the fourth order ordinary differential equation

$$u_{xxxx} + \frac{5}{3}uu_{xx} + \frac{5}{6}(u_x)^2 + \frac{5}{18}\left(x - tu + u^3\right) = 0, \tag{1.3}$$

which had been studied for $t = 0$ in [8] and [9] in connection with evaluating nonperturbative string effects in two-dimensional quantum gravity (Eq. (1.3) belongs to a class of massive string equations). In [10] the solution of

$$\left(W^3 - WW_{XX} - \frac{1}{2}(W_X)^2 + \frac{1}{10}W_{XXXX}\right) + \frac{15}{32}T\left(W^2 - \frac{1}{3}W_{XX}\right) = X$$

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with asymptotics $\sqrt[3]{X}$ as $X \to \pm\infty$ was treated numerically in connection with problems of quantum gravity. One can show that this solution $U(t, X)$ is also equivalent to the GP special solution of (1.1) for $t \geqslant 0$ (but not for $t < 0$).

Dubrovin showed in [11] and [12] directly by means of the theory of approximate symmetries [13] that it is the solution of (1.3) with asymptotics (1.2) that appears near the points of wave overturning for the very diverse singular dispersion perturbations of the equations of one-dimensional hydrodynamics.

The results of numerical simulations presented in [10] demonstrate rather strikingly that the GP special solution of the KdV equation possesses a domain of undamped oscillations for $t$ large enough. The authors of [10] did not conjecture any relation of their Letter to the GP special solution and raised the problem of describing this domain of oscillations. Meanwhile Gurevich and Pitaevskii [14] had used successfully the self-similar solutions of the averaged Whitham equations [15] to solve the problem.

The self-similar solutions in question were constructed in explicit form by Potemin [16]. However, the problem on the leading term of asymptotics of the GP special solution in the domain of Whitham oscillations has been open up to now. One not simple question still unanswered has been that on the phase shift.

Our purpose is to show how it is possible to construct the leading term of the GP special solution in the zone of oscillations without using any averaging methods. To this end we derive certain algebraic equations for the slowly varying amplitude and the leading term of the phase, which are actually equivalent to those of [16]. Moreover, we determine the phase shift of the solution in the oscillation zone.

Our approach may also be of use for the study of undamped oscillations of other common solutions to integrable partial and ordinary differential equations which are of importance in physics. In particular, it applies to two universal solutions of the KdV equation treated in the recent article [17]. Almost one problem in the approach is some awkwardness of analytical calculations. However, invoking modern programs for symbol calculations (in this work we use Maple) often allows one to get rid of such problems without particular difficulties.

## 2. Evaluation of phase shift

Consider the solution of the KdV equation that, for $t \to -\infty$ and $x \to \pm\infty$, is determined in the main from the cubic equation (1.2). It is known that for this solution for positive $t$ there is a domain where dissipationless shock waves appear.

We are aimed at constructing asymptotics of the solution in this domain, when $t \to \infty$. Following familiar techniques, we change the variables by

$$u = \sqrt{|t|}\,U(t, z),$$
$$z = \frac{x}{|t|^{3/2}}.$$

Then Eqs. (1.1) and (1.3) take the form

$$tU_t + \frac{1}{2}(U - 3zU_z) + UU_z + t^{-7/2}U_{zzz} = 0,$$
$$t^{-7}U_{zzzz} + \frac{5}{18}t^{-7/2}\big(6UU_{zz} + 3(U_z)^2\big) + \frac{5}{18}\big(z - U + U^3\big) = 0. \tag{2.1}$$

We now look for a solution $U$ of the system in the form of asymptotic series

$$U = U_0(\varphi, z) + t^{-7/4}U_1(\varphi, z) + t^{-7/2}U_2(\varphi, z) + \cdots, \tag{2.2}$$

where $U_0$, $U_1$ and $U_2$ are $2\pi$-periodic in the fast variable $\varphi$. This latter is assumed to be of the form

$$\varphi = t^{-7/4}f(z) + s(z),$$

where by $s(z)$ is meant precisely the phase shift.

For the unknown function $U_0$ we get the nonlinear system

$$Q^3\partial_\varphi^3 U_0 + QR\partial_\varphi U_0 + QU_0\partial_\varphi U_0 = 0,$$
$$Q^4\partial_\varphi^4 U_0 + \frac{5}{6}Q^2\big(2U_0\partial_\varphi^2 U_0 + (\partial_\varphi U_0)^2\big) + \frac{5}{18}\big(z - U_0 + U_0^3\big) = 0,$$

while the systems for $U_1$

$$Q^3\partial_\varphi^3 U_1 + Q(R + U_0)\partial_\varphi U_1 + Q\partial_\varphi U_0 U_1 = F_1,$$
$$Q^4\partial_\varphi^4 U_1 + \frac{5}{3}Q^2\big(U_0\partial_\varphi^2 U_1 + \partial_\varphi U_0\partial_\varphi U_1\big) + \frac{5}{18}\big(3U_0^2 + 3Q^2\partial_\varphi^2 U_0 - 1\big)U_1 = F_2,$$

and for $U_2$

$$Q^3\partial_\varphi^3 U_2 + Q(R + U_0)\partial_\varphi U_2 + Q\partial_\varphi U_0 U_2 = G_1,$$
$$Q^4\partial_\varphi^4 U_2 + \frac{5}{3}Q^2\big(U_0\partial_\varphi^2 U_2 + \partial_\varphi U_0\partial_\varphi U_2\big) + \frac{5}{18}\big(3U_0^2 + 3Q^2\partial_\varphi^2 U_0 - 1\big)U_2 = G_2$$

proves to be linear. Here, $F_1$ and $F_2$ are explicit functions depending on $z$ and $U_0$, and $G_1$ and $G_2$ are explicit functions depending on $z$ and $U_0$, $U_1$, i.e., the right-hand sides are explicit functions depending on $z$ and on the preceding corrections. We write

$$Q = f',$$
$$R = \frac{7}{4}\frac{f}{f'} - \frac{3}{2}z \tag{2.3}$$

for short.

From the compatibility condition of the equations for $U_0$ we obtain a first order equation

$$Q^2(\partial_\varphi U_0)^2 + \frac{1}{3}U_0^3 + RU_0^2 + \frac{1}{3}\big(18R^2 - 5\big)U_0 + \frac{1}{3}\big(15R - 54R^3 - 5z\big) = 0. \tag{2.4}$$

From the compatibility condition of the equations for $U_1$ we derive a nonlinear equation

$$\frac{d}{dz}R = \frac{1}{9}\frac{486R^4 - 171R^2 + 9zR + 5}{(54R^3 - 9R + z)(2R + 3z)} \tag{2.5}$$

for the unknown function $R = R(z)$. (In Section 4 we show that this equation agrees with results obtained earlier.) When requiring the compatibility of the equations for $U_2$, we deduce that the function $U_0(\varphi, z)$ should satisfy, together with (2.4), a nonlinear ordinary differential equation in the variable $z$ of the form

$$\partial_z^2 U_0 - \frac{\partial_\varphi^2 U_0}{(\partial_\varphi U_0)^2}(\partial_z U_0)^2 + \frac{P_3(U_0)}{(\partial_\varphi U_0)^2}\partial_z U_0 + \partial_\varphi U_0\big(s'' + Hs'\big) + \frac{P_4(U_0)}{(\partial_\varphi U_0)^2} = 0. \tag{2.6}$$

Here, $P_3(U_0)$ and $P_4(U_0)$ are polynomials in $U_0$ of degrees 3 and 4, respectively, with coefficients depending on $z$ and $R$. The function $H = H(z, R)$ is given by