Physics Letters A 374 (2010) 1464–1468

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Network synchronization in a population of star-coupled fractional nonlinear oscillators

Junwei Wang [a,*], Yanbin Zhang [b]

[a] School of Informatics, Guangdong University of Foreign Studies, Guangzhou 510006, PR China
[b] School of Computer Science, Hangzhou Dianzi University, Hangzhou 310018, PR China

ARTICLE INFO

Article history:
Received 3 December 2009
Received in revised form 18 January 2010
Accepted 21 January 2010
Available online 5 February 2010
Communicated by A.R. Bishop

Keywords:
Fractional-order system
Caputo fractional derivative
Star network
Chaos synchronization

ABSTRACT

The topic of fractional calculus is enjoying growing interest among mathematicians, physicists and engineers in recent years. For complex network consisting of more than two fractional-order systems, however, it is difficult to establish its synchronization behavior. In this Letter, we study the synchronized motions in a star network of coupled fractional-order systems in which the major element is coupled to each of the noninteracting individual elements. On the basis of the stability theory of linear fractional-order differential equations, we derive a sufficient condition for the stability of the synchronization behavior in such a network. Furthermore, we verify our theoretical results by numerical simulations of star-coupled network with fractional-order chaotic nodes.

1. Introduction

Fractional calculus (FC) is a generalization of ordinary (integer-order) differentiation and integration to its fractional (non-integer) order counterpart, which has been known since the letter between Leibniz and L'Hôpital on September 30, 1695. For the past 300 years, the theory of factional derivatives was developed primarily as a pure theoretical field useful only for mathematicians [1–3]. Nowadays, however, it was found that many systems in interdisciplinary fields, such as electrode–electrolyte polarization [4], viscoelastic systems [5], dielectric polarization [6], electromagnetic wave [7], and boundary layer effects in ducts [8], can be described by fractional differential equations (FDEs). These research efforts have shown that fractional derivatives provide an excellent tool for describing the memory and hereditary properties of various materials and processes.

It has been shown that some fractional-order dynamical systems, as generalizations of many well-known integer-order systems, can also behave chaotically, for example, the fractional-order Duffing system [9], the fractional-order Chua system [10], the fractional-order Lorenz system [11], the fractional-order Rössler system [12], the fractional-order Chen system [13], the fractional-order Lü system [14], the fractional-order unified system [15], and so on. Recently, due to its potential applications in secure communication and control processing, synchronization of fractional-order chaotic systems has received a great deal of attention [15–24]. Li et al. [16] numerically investigated the master–slave synchronization of fractional Chua and Rössler systems which represents the first report on the synchronization of fractional-order dynamical systems. In Ref. [17], Lu presented a drive-response synchronization method with linear output error feedback for a class of fractional chaotic systems via a scalar transmitted signal. A nonlinear controller for synchronizing integer-order differential systems has been successfully extended to fractional Chen system to achieve complete synchronization [18]. In Ref. [20], chaos synchronization of fractional Duffing, Lorenz and Rössler systems was studied with three different coupling methods. Using the pole placement technique, a nonlinear state observer has been designed for synchronizing a class of nonlinear fractional-order systems [23]. More recently, we proposed a new synchronization approach for coupled fractional-order systems based on the open-plus-closed-loop control method [24].

It should be noted that most of research efforts mentioned above have concentrated on studying synchronization behavior of two coupled fractional-order nonlinear oscillators. However, many systems in the real world usually consist of a large number of highly interconnected dynamical units to form a 'complex network' [25]. Examples of networked systems include both naturally occurring networks (such as ecological networks, food webs, cellular and metabolic networks, gene regulatory networks), as well as man-made networks (such as the World-Wide Web, electrical power-grids, collaboration networks of research scientists, social networks of acquaintances and friendships, supervisory control and communication networks, and distributed sensor networks). There-

* Corresponding author. Tel.: +86 20 39328577.
E-mail address: wangjunweilj@yahoo.com.cn (J.W. Wang).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2010.01.042

fore, it is very necessary to synchronize complex networks of coupled integer-order or fractional-order nonlinear oscillators. Because the nodes in complex networks are high-dimensional, the network connectivity is highly coupled across the population and the wiring can change over time, understanding the synchronization behavior in network organization is a difficult task and hence received less attention in the research field of fractional-order systems [26,27].

Our interest in this Letter is the analytical study of the synchronization phenomenon in complex networks of fractional-order nonlinear oscillators. Specifically, we will investigate the synchronization dynamics of a star network composed of N-coupled fractional-order systems. After a brief overview of fractional derivatives, the topology of star-coupled fractional-order systems is presented. Then by utilizing the stability theory of linear FDEs, we derive a stability criterion which guarantees, if satisfied, that the ensemble reaches complete network synchronization. Finally, we provide numerical simulations to illustrate the effectiveness of the theoretical analysis.

2. Preliminaries

2.1. Fractional derivatives

At present, there are many ways to define a fractional differential operator [1–3]. Here we adopt the most common one of them:

$$D_*^q f(t) = J^{m-q} f^m(t), \tag{1}$$

where $m := \lceil q \rceil$ is just the value q around up to the nearest integer, i.e., $m-1 < q < m$. Here $f^{(m)}$ is the ordinary mth derivative of f, and J^μ is the Riemann–Liouville integral operator of order $\mu > 0$, defined by

$$J^\mu g(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} g(\tau)\, d\tau, \tag{2}$$

where $\Gamma(\cdot)$ denotes the gamma function. It is common practice to call the operator D_*^q the "q-order Caputo differential operator", which has emerged as an useful tool for modeling many phenomena in physics and engineering. For the range of q, a particularly important case in many engineering applications is $0 < q < 1$. In this situation, (1) together with (2) reduced to

$$\frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(1-q)} \int_0^t (t-\tau)^{-q} f'(\tau)\, d\tau, \tag{3}$$

where $\frac{d^q f(t)}{dt^q} \triangleq D_*^q f(t)$, which will be used throughout this Letter.

According to the classical theory of ordinary differential equations (ODEs), we need to specify initial conditions to make sure that the solution is unique. The Caputo fractional derivative is considered in this Letter because it allows initial conditions given in terms of integer-order derivatives, which represent well-understood features of a physical situation and therefore their values can be measured accurately.

2.2. Stability theory of fractional differential equations

Now, we present a stability theorem for the incommensurate fractional-order systems which will be used in the next section. Consider the following n-dimensional linear fractional-order system:

$$\begin{cases} \dfrac{d^{q_1} x_1}{dt^{q_1}} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ \dfrac{d^{q_2} x_2}{dt^{q_2}} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\ \vdots \\ \dfrac{d^{q_n} x_n}{dt^{q_n}} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n, \end{cases} \tag{4}$$

where all q_i $(i = 1, 2, \ldots, n)$ are rational numbers between 0 and 1. Assume M is the lowest common multiple of the denominators u_i of q_i, where $q_i = v_i/u_i$, $(u_i, v_i) = 1$, $u_i, v_i \in Z^+$, for $i = 1, 2, \ldots, n$. Define

$$\Delta(\lambda) = \begin{pmatrix} \lambda^{Mq_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{Mq_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda^{Mq_n} - a_{nn} \end{pmatrix}. \tag{5}$$

Lemma 1. *(See [28].) The zero solution of system (4) is globally asymptotically stable in the Lyapunov sense if all roots λ of the equation* $\det(\Delta(\lambda)) = 0$ *satisfy* $|\arg(\lambda)| > \pi/2M$.

3. Synchronization of star-coupled fractional-order systems

Throughout this Letter, we will denote scalar variables in lower case, and vectors (or vector-valued functions) in bold-type lower case, and matrices in bold-type upper case. We consider an ensemble of N cells, coupled through a major element, with each cell being an m-dimensional system obeying the standard fractional kinetic equations:

$$\frac{d^{\boldsymbol{q}} \boldsymbol{x}}{dt^{\boldsymbol{q}}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x}), \tag{6}$$

where $\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T \in \boldsymbol{R}^m$ is the state variable, $\boldsymbol{A} \in \boldsymbol{R}^{m\times m}$ is a constant matrix, and $\boldsymbol{h} : \boldsymbol{R}^m \to \boldsymbol{R}^m$ defines a continuously vector-valued function. With the fractional orders $\boldsymbol{q} = (q_1, q_2, \ldots, q_m)$, we define

$$\frac{d^{\boldsymbol{q}} \boldsymbol{x}(t)}{dt^{\boldsymbol{q}}} = \left(\frac{d^{q_1} x_1(t)}{dt^{q_1}}, \frac{d^{q_2} x_2(t)}{dt^{q_2}}, \ldots, \frac{d^{q_m} x_m(t)}{dt^{q_m}} \right)^T. \tag{7}$$

Many fractional-order chaotic systems can be described by system (6), such as fractional-order Lorenz system [11], fractional-order Rössler system [12], fractional-order Chen system [13], and fractional-order unified system [15], just to mention a few. Assume that function $\boldsymbol{g}$ is sufficiently differentiable, Eq. (6) has a continuous, differential and bounded solution and its derivative function is also bounded for some fractional orders [29–31].

The entire network is a system of $m(N+1)$ fractional differential equations. In particular, the state equations are

$$\begin{cases} \dfrac{d^{\boldsymbol{q}} \boldsymbol{x}_1}{dt^{\boldsymbol{q}}} = \boldsymbol{A}\boldsymbol{x}_1 + \boldsymbol{h}(\boldsymbol{x}_1) + \boldsymbol{K}_1(\boldsymbol{z} - \boldsymbol{x}_1), \\ \dfrac{d^{\boldsymbol{q}} \boldsymbol{x}_2}{dt^{\boldsymbol{q}}} = \boldsymbol{A}\boldsymbol{x}_2 + \boldsymbol{h}(\boldsymbol{x}_2) + \boldsymbol{K}_2(\boldsymbol{z} - \boldsymbol{x}_2), \\ \vdots \\ \dfrac{d^{\boldsymbol{q}} \boldsymbol{x}_N}{dt^{\boldsymbol{q}}} = \boldsymbol{A}\boldsymbol{x}_N + \boldsymbol{h}(\boldsymbol{x}_N) + \boldsymbol{K}_N(\boldsymbol{z} - \boldsymbol{x}_N), \\ \dfrac{d^{\boldsymbol{q}} \boldsymbol{z}}{dt^{\boldsymbol{q}}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{h}(\boldsymbol{z}) + \sum_{j=1}^{N} \left[\boldsymbol{K}_j(\boldsymbol{x}_j - \boldsymbol{z})\right]. \end{cases} \tag{8}$$

The system (8) consists of a population of N identical noninteracting elements $\boldsymbol{x}_i$ and of the major element $\boldsymbol{z}$, which is coupled to each of the elements $\boldsymbol{x}_i$ with the coupling strength $\boldsymbol{K}_i \in \boldsymbol{R}^{m\times m}$.