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Reconstruction of driving forces through recurrence plots

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ABSTRACT

We consider the problem of reconstructing one-dimensional driving forces only from the observations of driven systems. We extend the approach presented in a seminal paper [M.C. Casdagli, Physica D 108 (1997) 12] and propose a method that is robust and has wider applicability. By reinterpreting the work of Thiel et al. [M. Thiel, M.C. Romano, J. Kurths, Phys. Lett. A 330 (2004) 343], we formulate the reconstruction problem as a combinatorial optimization problem and relax conditions by assuming that a driving force is continuous. The method is demonstrated by using a tent map driven by an external force.

1. Introduction

Nonstationary time series are ubiquitous in science and technology. Recurrence plots (RPs) [1,2] of nonstationary time series provide useful information on time series [3–7].

One of the common models of nonstationary systems is a deterministic dynamical system driven by a slowly varying parameter [3,8]. In this model, usually the slowly varying parameter is not known and has to be reconstructed from the observations of the driven system. Casdagli [3] has shown that in a sufficiently large embedding space, the RP of the driven system becomes similar to that of the slowly varying parameter. He has also reconstructed the waveform of the slowly varying parameter on the basis of the fact that a part of the RP of a driven system matches the waveform of its slowly varying parameter under some conditions. Therefore, we can interpret that Casdagli has divided the problem for reconstructing the slowly varying parameter into two parts: the approximation of the RP of the slowly varying parameter and the reconstruction of the original signal from the RP. Although the approximation part is quite generic, the reconstruction part is imperfect because the slowly varying parameter should take the global maximum at the first local maximum.

There is another solution for the reconstruction of the original signal from the RP. Thiel et al. [9] has shown that the original signal can be reconstructed from an RP if the RP is obtained by considering a one-dimensional embedding space. If we combine the solution obtained by Casdagli by approximation with the method proposed by Thiel et al., we may be able to reconstruct the original signal in various problems. However, the method proposed by Thiel et al. is applicable only for RPs generated cleanly from a time series. In our study, the method proposed by Thiel et al. is not applicable since the RP is obtained by approximation.

In this study, we propose a method to reconstruct the original signal from its RP that is not generated cleanly from a time series. First, we reinterpret the method of Thiel et al. as a problem of combinatorial optimization. Then, assuming the slowly varying parameter to be continuous, we modify the objective function of the combinatorial optimization in two ways. We show that, by using the proposed methods, it is possible to reconstruct a wider variety of slowly varying driving forces.

The rest of this Letter is organized as follows. In Section 2, we introduce a method for reconstructing a driving force from an RP. In Section 3, we formulate the combinatorial optimization prob-

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lem for reconstructing the original signal from its RP. In Section 4, we propose two objective functions for cases where RPs are not generated cleanly from time series. In Section 5, we present an example of a tent map driven by a one-dimensional driving force and also discuss the effects of changing the parameters in the objective functions. In Section 6, we present the conclusions of this study.

2. Recurrence plots of externally driven systems

2.1. Dynamical systems driven by external forces

A large number of nonlinear phenomena are observed in the world. For modeling them, sometimes dynamical systems driven by slowly varying external forces are considered. These systems may be defined by

$$s_{i+1} = f(s_i, \gamma_i), \qquad x_i = h(s_i), \tag{1}$$

where s_i is a state of the system, γ_i is a slowly varying driving force, and x_i is an observation at time i.

We assume that $\{x_i\}$ can be observed, but not $\{\gamma_i\}$. Then we need to estimate $\{\gamma_i\}$ from $\{x_i\}$.

2.2. Recurrence plots

Recurrence plots (RPs) [1,2] are used for visualizing the correlations within a time series. In order to generate an RP from a time series $\{x_i\}_{i=1,\dots,N}$ of length N, first, we consider a vector v_i defined by

$$v_i = (x_i, x_{i+\tau}, \dots, x_{i+(m-1)\tau}), \quad i = 1, \dots, N_m, \tag{2}$$

where τ is the time delay, m is the embedding dimension, and $N_m = N - (m-1)\tau$. In this study, we set $\tau = 1$.

Next, we consider a two-dimensional plane in which two axes represent time. In this plane, a point exists at (i, j) if $\|v_i - v_j\| \leqslant r$. The distribution of the points in this plane gives the RP. We express this distribution of points by

$$R(i, j) = \begin{cases} 1, & \text{when } \|v_i - v_j\| \leqslant r, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

When $R(i, j) = 1$, there exists a point at (i, j) in the plane. When $R(i, j) = 0$, no such point exists in the plane. In this study, we use the maximum norm for the calculation of $\|v_i - v_j\|$.

There are some studies in which RPs are quantified [10–15]. We can estimate dynamical invariants such as correlation entropy and correlation dimension from RPs [12,14]. Thus, it appears that RPs can provide a large amount of information on the original time series. In fact, it is known that the original time series can be reconstructed from their RPs under certain conditions [9,16]. In this study, we propose reconstruction methods that can be used under relaxed conditions by assuming that the original time series are continuous. The details of these methods are discussed in Sections 3 and 4.

2.3. Estimation of recurrence rate

Casdagli [3] has shown that the RP of a driving force can be approximated from the RP from the driven system. Since his study forms an important foundation of our study, we briefly summarize the result of this study.

First, we consider the density of all the points in an RP, which is called the recurrence rate $C_m(r)$ and given by

$$C_m(r) = \frac{2}{N_m(N_m - 1)} \sum_{i=1}^{N_m} \sum_{j=1}^{i-1} \Theta\bigl(r - \|v_i - v_j\|\bigr), \tag{4}$$

where Θ is the Heaviside step function.

Among the points in an RP, the points at (i, j) such that the distance between γ_i and γ_j is larger than the threshold $\epsilon > 0$ are undesirable in the approximation of the RP of the driving force. The density of these undesirable points, which is called the false recurrence rate $P_m^{\mathrm{F}}(r, \epsilon)$, is given by

$$P_m^{\mathrm{F}}(r, \epsilon) = C_m(r) - \frac{2}{N_m(N_m - 1)} \times \sum_{i=1}^{N_m} \sum_{j=1}^{i-1} \Theta\bigl(r - \|v_i - v_j\|\bigr) \Theta\bigl(\epsilon - \|\gamma_i - \gamma_j\|\bigr). \tag{5}$$

Then, under some genericity conditions, $C_m(r)$ and $P_m^{\mathrm{F}}(r, \epsilon)$ obey the following scaling laws:

$$C_m(r) = \begin{cases} \mathrm{O}(r^m), & m \leqslant D, \\ \mathrm{O}(r^D), & m > D, \end{cases} \tag{6}$$

$$P_m^{\mathrm{F}}(r, \epsilon) \leqslant \begin{cases} \mathrm{O}(r^m), & m \leqslant 2D - 1, \\ \mathrm{O}(r^{2D-1}), & m > 2D - 1, \end{cases} \tag{7}$$

respectively, where D denotes the fractal dimension of the set $\{(s_i, \gamma_i)\}$.

If we assume m to be larger than D, it follows from the scaling laws that $P_m^{\mathrm{F}}(r, \epsilon)/C_m(r) \leqslant \mathrm{O}(r^{\min(m-D, D-1)})$. Therefore, for sufficiently small r, $P_m^{\mathrm{F}}(r, \epsilon)/C_m(r) \approx 0$. This condition means that undesirable points vanish faster than the desirable points. Thus, the RP constructed from the observations of a driven system can be regarded as an approximation of the RP of the driving force.

3. Reconstruction of time series from recurrence plots

As explained in the previous section, the RP of a driving force can be approximated from the RP of the driven system. If a good approximation is obtained, then the next step is to reconstruct the driving force from the approximated RP.

In this section, we consider a method to reconstruct a time series from its RP.

3.1. Formulation of problem

The reconstruction of the values in the original time series is essentially impossible because RPs do not contain information on the values in the original time series.

Hence, in this study, we reconstruct a rank order ρ of the time series $\{x_i\}_{i=1,\dots,N}$. The rank order is defined as a permutation on $\{1, \dots, N\}$ that sorts the time series in the non-decreasing order, namely, $x_{\rho^{-1}(1)} \leqslant \dots \leqslant x_{\rho^{-1}(N)}$. In other words, this order means that each x_i in $\{x_i, \ i = 1, \dots, N\}$ is the $\rho(i)$th smallest element in the set. Thus, the time series $\rho(1), \dots, \rho(N)$ roughly resembles the time series $x_1, \dots, x_N$ such that $\{\rho(i)\}$ can be regarded to be obtained using a monotonic function from the time series $\{x_i\}$. Additionally, we introduce an inverse rank order, which sorts the time series in the non-increasing order, for ideal reconstruction, because it is indistinguishable from the normal rank order if only the information contained in an RP is considered.

Therefore, our problem is to find a permutation σ that satisfies either $x_{\sigma^{-1}(1)} \leqslant \dots \leqslant x_{\sigma^{-1}(N)}$ or $x_{\sigma^{-1}(1)} \geqslant \dots \geqslant x_{\sigma^{-1}(N)}$ by using only the information contained in an RP. To find σ, we have to make some assumptions with respect to the generation of R from the time series $\{x_i\}$. First, we assume that R is cleanly generated from $\{x_i\}$, exactly in the manner defined in Section 2.2. However, it should be noted that in our case only an approximate RP is obtained; hence this assumption may not hold in our case.