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Geometric phases and entanglement of two qubits with XY type interaction

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ABSTRACT

It is shown that geometric phases and entanglement may fail to detect level crossings for two qubits with XY interaction. The rotating magnetic field produces a magnetic monopole sphere like conducting spheres in that only a ground state evolving adiabatically outside the sphere acquires a geometric phase.

Energy is the most primary quantity determining the properties of physical systems. When energy levels are crossing or avoided crossing as system parameters vary, a quantum system exhibits rich physics. For example, if two energy levels, initially separated, become close but not crossing and then far away again, then the non-adiabatic Landau–Zener transition between them takes place [1]. In molecular systems, the conical intersection of adiabatic electronic states play key role in radiationless chemical reaction [2]. A quantum state traveling adiabatically around level crossing points accumulates the geometric phase in addition to the dynamical phase [3,4]. Berry provided a beautiful interpretation of the geometric phase as the magnetic flux due to magnetic monopoles located at the level crossing points. A quantum phase transition, a dramatic change in the ground state driven by parameters in the zero temperature, is associated with level crossings or avoided crossings between the ground and exited energy levels [5].

When levels are crossings or avoided crossing as the parameters of a Hamiltonian varies, the corresponding quantum states can change significantly. So, it is important to know what physical quantities could be good indicators to level crossings or avoided crossings. Recently, the relation between geometric phases, entanglement, and level crossings for the atomic Breit–Rabi Hamiltonian has been investigated by Oh et al. [6]. The adiabatic geometric phase is a direct way to detect level crossings because it is associated with level crossing or avoided crossing. The fidelity could be a good indicator to level crossings because it measures the distance between two quantum states. Also entanglement, referring to quantum correlation between subsystems, could be a good indicator to level crossings because entanglement measures tell us whether a quantum state is separable or not.

In this Letter, we study a system of two qubits with the XY type interaction as a minimal model showing the geometric phase, the entanglement jump, and the abrupt change in the fidelity at level crossings. It is demonstrated that entanglement jump may fail to capture level crossings. The geometric phase is shown to be zero for the glancing level crossings. Also it is shown that the magnetic monopole charge producing the geometric phase is distributed on the surface of the sphere like a conducting sphere of electric charges.

Let us consider a simple system of two qubits with the XY type interaction. As shown later, it contains rich physics in spite of its simplicity. The Hamiltonian of the system reads

$$H(\lambda,\gamma) = -\frac{(1+\gamma)}{2}\sigma_1^x\sigma_2^x - \frac{(1-\gamma)}{2}\sigma_1^y\sigma_2^y - \frac{\lambda}{2}\left(\sigma_1^z+\sigma_2^z\right), \qquad (1)$$

where γ is an anisotropy factor, λ is an external magnetic field in the z direction, and σ_i^α are the Pauli matrices of the ith qubit with $\alpha = x, y, z$. The Hamiltonian (1) is the simplest form of an spin 1/2 XY chain in a transverse magnetic field (hereafter called simply the XY model), which is exactly solvable [7] and becomes a paradigmatic example in the study of quantum phase transitions.

The eigenvalues and eigenstates of Hamiltonian (1) can be easily obtained by rewriting (1) in the matrix form,

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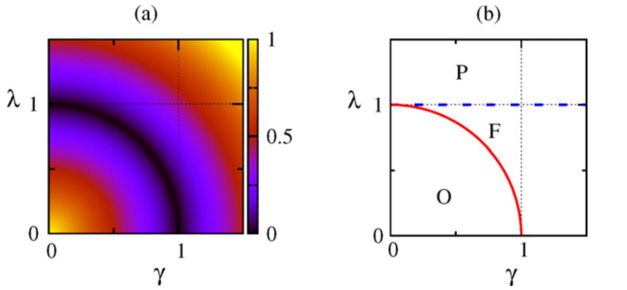

Fig. 1. (Color online.) (a) Energy gap between the ground and first excited states as a function of λ and γ for two qubits with XY type interaction. The level crossing (dark line) takes places on the circle $\lambda^2+\gamma^2=1$. (b) The ground state phase diagram of the XY model. P denotes the paramagnetic phase, F the ordered ferromagnetic phase, and O the oscillatory phase [8].

$$H(\lambda,\gamma)=-\begin{pmatrix}\lambda & 0 & 0 & \gamma\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ \gamma & 0 & 0 & -\lambda\end{pmatrix}=H_{\text{even}}+H_{\text{odd}}. \tag{2}$$

The Hamiltonian $H_{\text{even}}=-\left(\begin{smallmatrix}\lambda & \gamma\\ \gamma & -\lambda\end{smallmatrix}\right)$ is defined on the subspace spanned by $\{|00\rangle,|11\rangle\}$. This looks like a Hamiltonian of a spin 1/2 in a magnetic field. It is easy to write down the eigenvalues and eigenvectors of H_{even}

$$E^e_{\pm}=\pm\sqrt{\lambda^2+\gamma^2}, \tag{3a}$$

$$\left|E^e_-\right\rangle=\cos\frac{\theta}{2}|00\rangle+\sin\frac{\theta}{2}|11\rangle, \tag{3b}$$

$$\left|E^e_+\right\rangle=-\sin\frac{\theta}{2}|00\rangle+\cos\frac{\theta}{2}|11\rangle, \tag{3c}$$

where $\tan\theta\equiv\gamma/\lambda$. On the other hand, the Hamiltonian $H_{\text{odd}}=-\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ defined on the subspace of $\{|01\rangle,|10\rangle\}$ has the eigenvalues $E^o_{\pm}=\pm 1$ and the eigenvectors

$$\left|E^o_{\pm}\right\rangle=\frac{1}{\sqrt{2}}\big(|01\rangle\mp|10\rangle\big). \tag{4}$$

Let us look at where the level crossings between the ground and first exited states are located in the parameter space of γ and λ. The condition of level crossing is $E^o_-=E^e_-$. As shown in Fig. 1, this is just a circle

$$\lambda^2+\gamma^2=1. \tag{5}$$

Surprisingly, this is identical to the disorder line of the XY model, which separates the ordered oscillating phase in the region $\lambda^2+\gamma^2<1$ from the ferromagnetic phase [8].

It is also remarkable that the ground state and the ground-state energy of the XY model are similar in forms to Eqs. (3a) and (3b), respectively. One may wonder why the phase diagram of two qubits with XY type interaction looks like that of the XY model in thermodynamic limit, as depicted in Fig. 1. This could be explained by the fact that the ground state energy E_0 of a system of N identical particles with at most two particle interaction can be written as $E_0=\frac{N}{2}\sum_i\epsilon_i\langle\epsilon_i|D^2|\epsilon_i\rangle$ [9], where $D^2=\text{tr}_{3,\ldots,N}(|\Psi\rangle\langle\Psi|)$ is the two-particle reduced density matrix derived from the ground state $|\Psi\rangle$ satisfying $H|\Psi\rangle=E_0|\Psi\rangle$. And the reduced Hamiltonian $K\equiv H_1+H_2+(N-1)H_{12}$, derived from the full Hamiltonian $H=\sum_{i=1}^N H_i+\sum_{i<j}H_{ij}$, has eigenvalues ϵ_i and eigenstates $|\epsilon_i\rangle$. The XY model can be mapped to an one-dimensional spinless fermion system through the Jordan–Wigner transformation. Thus it could be understood why the eigenvalues and eigenstates of Hamiltonian (1) contain the partial information on the XY model [10].

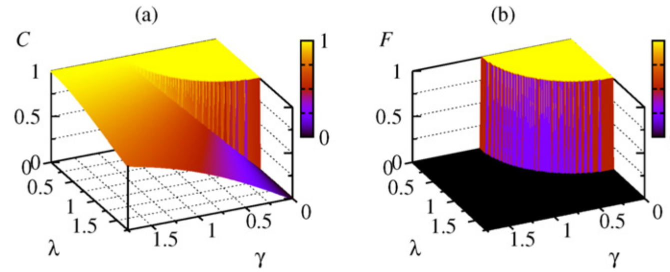

Fig. 2. (Color online.) (a) Concurrence $C(|\psi_0\rangle)$ of the ground state $|\psi_0\rangle$ and (b) fidelity F between $|E^0_-\rangle$ and $|\psi_0\rangle$ as a function of γ and λ.

Now, let us examine whether the entanglement is always a good indicator to level crossings [11]. For a pure two-qubit state $|\psi\rangle=a|00\rangle+b|01\rangle+c|10\rangle+d|11\rangle$ with $|a|^2+|b|^2+|c|^2+|d|^2=1$, a well-known entanglement measure is the concurrence $C(|\psi\rangle)=2|ad-bc|$. For the ground state $|\psi_0\rangle$ given by $|E^e_-\rangle$ and $|E^o_-\rangle$, it is written as

$$C\big(|\psi_0\rangle\big)=\begin{cases}\sin\theta, & \text{for } \gamma^2+\lambda^2>1,\\ 1, & \text{for } \gamma^2+\lambda^2<1.\end{cases} \tag{6}$$

As shown in Fig. 2(a), the entanglement changes abruptly as γ and λ passes across the circle, i.e., the disorder line. It seems that the entanglement works well as an indicator to quantum phase transitions. However, along the γ axis, i.e., $\theta=\pi/2$, the concurrence does not change even if the ground state changes from $|E^o_-\rangle$ to $|E^e_-\rangle$. This demonstrate that the entanglement measure may fail to capture a level crossing which happens between the ground states with same degrees of entanglement. As the number of particles increase, the dimension of a sub-Hilbert space whose states have the same degree of entanglement also increases. Thus it is not hard to imagine a quantum phase transition which occurs between the ground states with the same degree of entanglement. In this case, the entanglement is not a good indicator to quantum phase transitions.

The fidelity F between two quantum states $|\psi\rangle$ and $|\phi\rangle$, defined by $F\equiv|\langle\psi|\phi\rangle|^2$, is one of the useful measures of distance between two quantum states. It could be a good indicator to level crossings because the ground states before and after level crossings changes abruptly [12]. It is simple to calculate the fidelity $F=|\langle E^o_-|\psi_0\rangle|^2$ between the ground state $|\psi_0(\lambda,\gamma)\rangle$ and the reference state $|E^0_-\rangle$ as a function of γ and λ. As illustrated in Fig. 2(b), the fidelity jumps on the circle.

Let us turn to the relation between geometric phases and level crossings. The geometric phase of the XY model in connection with quantum phase transitions has been study in Refs. [13–15] where the degeneracy points are located on the XX line, i.e. along the λ axis, of the XY model. In contrast, the system here has the degeneracy points located on the circle. Let us rotate the Hamiltonian about the λ axis by angle ϕ, $\tilde{H}(\lambda,\gamma,\phi)=U_z^\dagger(\phi)H(\lambda,\gamma)U_z(\phi)$ with $U_z(\phi)\equiv\exp[-i\frac{\phi}{2}(\sigma_1^z+\sigma_2^z)]$. It is easy to obtain the transformed Hamiltonian

$$\tilde{H}(\lambda,\gamma,\phi)=-\begin{pmatrix}\lambda & 0 & 0 & \gamma e^{-i2\phi}\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ \gamma e^{i2\phi} & 0 & 0 & -\lambda\end{pmatrix}. \tag{7}$$

The comparison of Eqs. (2) and (7) shows two things, which gives rise to interesting consequences. First, $\tilde{H}_{\text{odd}}$ is independent of angle ϕ. Second, $\tilde{H}_{\text{even}}$ looks like that of a spin 1/2 particle in a magnetic field

$$\mathbf{B}=\sqrt{\lambda^2+\gamma^2}(\sin\theta\cos 2\phi,\sin\theta\sin 2\phi,\cos\theta). \tag{8}$$