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The effect of time-delay on anomalous phase synchronization

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ABSTRACT

Anomalous phase synchronization in nonidentical interacting oscillators is manifest as the increase of frequency disorder prior to synchronization. We show that this effect can be enhanced when a time-delay is included in the coupling. In systems of limit-cycle and chaotic oscillators we find that the regions of phase disorder and phase synchronization can be interwoven in the parameter space such that as a function of coupling or time-delay the system shows transitions from phase ordering to disorder and back.

1. Introduction

A population of nonidentical nonlinear oscillators will eventually synchronize when sufficiently strongly coupled in a suitable topology. In some cases it has been observed that the degree of disorder in the oscillator frequencies—as characterized by their variance, say—first increases with coupling before eventually decreasing. This phenomenon has been termed *anomalous* phase synchronization (APS) [1–4], which has a broad range of applications.

The emergence of synchrony in groups of interacting nonidentical oscillators is a phenomenon of considerable interest [5], with applications in a variety of contexts ranging from epidemiology to cellular biology. Phase synchronization (PS) [6]—as opposed to complete synchronization—arises naturally in many areas of physical sciences since the subsystems that are coupled can have different amplitudes and a range of internal time-scales.

How does global phase synchronization come about in such a population? The manner in which all the oscillators in a mutually interacting group eventually adopt a common frequency of oscillation is of considerable importance, and one which has been explored to some extent in earlier work [1–3]. A natural expectation might be that the approach to global synchronization is monotonic: namely that two of the oscillators synchronize, then three, and then gradually increasing numbers of oscillators mutually synchronize.

This expectation does not hold in systems that show the anomalous phase synchronization. The interaction among the different systems acts to first drive systems out of synchrony before the strength of the interaction eventually forces them to a common dynamics. The intermediate disorder can arise from a number of different sources—non-isochronicity [7], shear [8] or differences in other internal parameters [5]. The full generality of this phenomenon is not known, and thus the APS is of interest both from a conceptual as well as an applications point of view.

In the present work we study the process of phase-synchronization in oscillators with time-delayed coupling. Time-delay is both natural and inevitable when considering interactions among systems that are spatially separated. From a mathematical point of view, time-delay makes the dynamical system effectively infinite-dimensional: this can open up a range of time-scales, interactions, and novel dynamical behaviour such as amplitude-death [9–12] and the phase-flip bifurcation [12,13]. In addition, delay offers an additional parameter that can be varied, and if exogenous, can provide a suitable means of effecting control. The implications of APS in such systems is therefore of considerable interest.

Our main result here is that APS can be enhanced with time-delay: the degree of initial disorder may be significantly larger than for the zero time-delay (or instantaneous coupling) case [1,2]. We also find that in situations when there is no APS, delay coupling can cause APS to occur. Using the delay or the coupling

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strength as a parameter, systems which are in phase synchrony can be driven out of synchrony and back again into phase synchrony. The regions of parameter space corresponding to desynchronized dynamics are interspersed among those corresponding to the synchronized phase, and thus the anomalous behaviour can be manifest as a transition from PS to phase disorder, and back to PS.

Some insight can be obtained from a study of the simplest situation, namely of two nonidentical delay coupled oscillators and we treat this case in the next section. The specific models we examine are the Landau–Stuart system, where the dynamics can be on limit cycles, and a somewhat more complex food-web model studied in [1,2], where APS was observed. Globally coupled oscillators are studied in Section 3. In all these examples, our results indicate that upon inclusion of time-delay, the region of APS can be enlarged, and the degree of disorder can be enhanced. The Letter concludes with a brief summary in Section 4.

2. Anomalous synchronization in two nonidentical oscillators

2.1. The Landau–Stuart system

Consider the case of two delay-coupled limit cycle Landau–Stuart oscillators [11,12]:

$$\dot{Z}_1(t) = \left(1 + i\Omega_1 - |Z_1(t)|^2\right)Z_1(t) + \epsilon\left[Z_2(t-\tau) - Z_1(t)\right],$$
$$\dot{Z}_2(t) = \left(1 + i\Omega_2 - |Z_2(t)|^2\right)Z_2(t) + \epsilon\left[Z_1(t-\tau) - Z_2(t)\right]. \quad (1)$$

The variables $Z_j(t)$ are the complex amplitudes of the oscillators, $|Z_j| = 1$ is the attracting limit cycle, Ω_j are the corresponding frequencies in absence of coupling, and ϵ is the coupling strength. We consider here strongly mismatched oscillators with $\Omega_1 = 4$ and $\Omega_2 = 7$ ($\Delta\Omega = 3$). For the case of instantaneous coupling, $\tau = 0$, the effective averaged frequencies $\omega_{1,2}$ of the two oscillators are plotted in Fig. 1(a) as a function of the coupling strength ϵ. The frequency difference $\Delta\omega$ decreases monotonically as shown in Fig. 1(b) and PS results above a critical coupling, $\epsilon \sim 1.5$. At these parameter values, APS does not occur: see Refs. [1,2] for details.

Introduction of a finite delay ($\tau \neq 0$) gives the phase diagram shown in Fig. 2; the oscillators are not in synchrony in the shaded region in the τ–ϵ plane and therefore there is the possibility of APS. Along a line of fixed coupling strength $\epsilon = 1$, say, examination of the largest few Lyapunov exponents [14] and the difference in frequency of the two oscillators as a function of τ reveals APS, as shown in Fig. 3(a), (b). As can be seen there is a finite range of time-delay for which the individual oscillator frequencies are different, and the difference $\Delta\omega$ can be larger than the initial $\Delta\Omega = 3$. Note that this region of anomaly is in effect caused by time-delay, since when $\tau = 0$, there is no region of APS (Fig. 1). Similar results have been also reported for phase only oscillators in Ref. [15] and subsequently in Ref. [16–19]. However, unlike the situation in Ref. [15], we do not find any evidence for hysteresis here: the curve in Fig. 3(b) is a composite of 50 simulations, each starting from different initial conditions (ruling out the possibility of multistability). Over a broad range of coupling parameters (results not shown here) there does not appear to be any hysteresis.

As is also evident in Fig. 2, APS can occur at fixed time-delay by variation of the coupling parameter ϵ. Note that the introduction of delay can reduce the onset of synchronization (see for small τ in Fig. 2). Results are shown in Fig. 3(c) and (d) for the Lyapunov exponentand frequency difference respectively at $\tau = 0.75$. Since this occurs with variation of the coupling strength for a fixed nonzero time-delay as well, the process is an example of APS arising from the time-delay interaction.

For the symmetric system when $\Omega_1 = \Omega_2$, there is no anomalous synchronization, but there is evidence for a phase-flip bifurcation [13] when the frequency increases in a manner similar to the

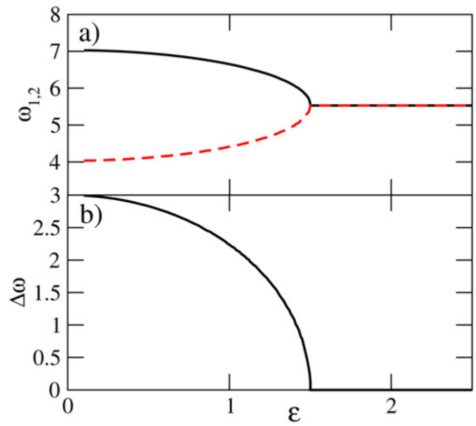

Fig. 1. In the Landau–Stuart system, Eq. (1) for the case of instantaneous coupling, $\tau = 0$, the variation of (a) the individual frequencies, $\omega_{1,2}$ of the two oscillators, and (b) their frequency difference, $\Delta\omega$ as a function of the coupling strength, ϵ.

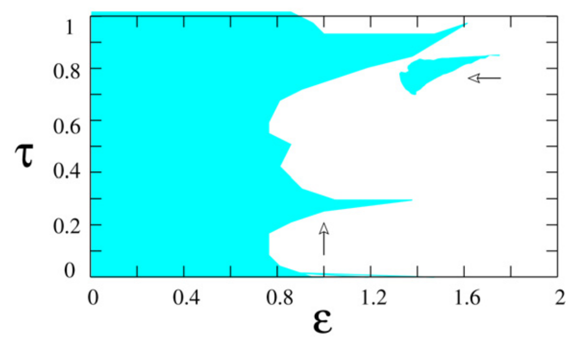

Fig. 2. Schematic phase diagram in the τ–ϵ plane for nonidentical Landau–Stuart oscillators, Eq. (1). The shaded region corresponds to desynchronized motion, and therefore APS can occur when parameter variation includes a path that traverses this region.

frequency increase shown in Fig. 3(b). This suggests that the phenomenon of anomalous synchronization may be the counterpart of the phase-flip in nonidentical systems. Since experimental verification of the phase-flip has been carried out in recent work [20], a systematic exploration of anomalous phase synchronization in a similar dynamical system (delay-coupled Chua oscillators) should be feasible.

Because of the nature of the boundary of the synchronized region, the anomalous effect can be evidenced as a transition from a synchronized state to a desynchronized state, and back to synchrony, as in Fig. 3(b), (d); see the parameter range marked by arrow A. In this system APS occurs when the dynamics is quasiperiodic ($\lambda_1 = \lambda_2 = 0$) but in general, the motion can even be chaotic, as in the example below.

2.2. Chaotic oscillators

We next analyze a model that has been studied earlier [1,2],

$$\dot{x}_{1,2}(t) = x_{1,2} - 1.5 - 0.1x_{1,2}y_{1,2},$$
$$\dot{y}_{1,2}(t) = -\beta_{1,2}y_{1,2} + 0.1x_{1,2}y_{1,2} - 0.6y_{1,2} + \epsilon\left[y_{2,1}(t-\tau) - y_{1,2}(t)\right],$$
$$\dot{z}_{1,2}(t) = -10z_{1,2} + 0.1 + 0.6y_{1,2}z_{1,2}. \quad (2)$$

This is a system of two coupled food-webs, each of which (differentiated by subscripts 1 or 2) describes a three level "vertical" food chain. The variables x correspond to the vegetation, which is