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Integrable motion of a vortex dipole in an axisymmetric flow

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ABSTRACT

The evolution of a self-propelling vortex dipole, embedded in an external nondivergent flow with constant potential vorticity, is studied in an equivalent-barotropic model commonly used in geophysical, astrophysical and plasma studies. In addition to the conservation of the Hamiltonian for an arbitrary point vortex dipole, it is found that the angular momentum is also conserved when the external flow is axisymmetric. This reduces the original four degrees of freedom to only two, so that the solution is expressed in quadratures. In particular, the scattering of antisymmetric dipoles approaching from the infinity is analyzed in the presence of an axisymmetric oceanic flow typical for the vicinity of isolated seamounts.

1. Introduction

Vortices have been recognized to be key elements in turbulent fluid motion at a wide range of scales. The Coriolis force on rotating planets or the Lorentz force due to magnetic field in plasmas makes large-scale flows anisotropic, quasi-two-dimensional (e.g., the horizontal velocity of oceanic currents is much larger than vertical velocity). Coherent, long-lived vortices and jets emerge naturally as a result of self-organization of turbulent motion observed in such anisotropic media; they are well identified by the wavelet transform and Okubo–Weiss criterion, e.g., [1]. Coherent vortices are very efficient in trapping passive tracers for long times and transporting them over anomalously large distances. For this reason, mutual interaction of isolated vortices, their stability and effects of background currents have been intensively studied in the last decades as summarized in a number of reviews [2–6].

Signatures of well-separated, nearly circular or elliptical, monopolar vortices of both signs are common in geophysical flows. Two vortices of opposite sign often form a self-propelling dipolar couple which provides anomalous transport of scalar properties for especially long distances. They are easily excited in two-dimensional flows and they appear to be the universal outcome of an external forcing possessing a nonzero linear momentum [7–9].

Many theoretical studies of vortex couples in geophysical flows and plasmas have been essentially constrained by the β-effect which allows the permanent form solutions only with zonal direction of propagation. In the traditional quasigeostrophic approximation such steadily propagating solutions (modons as labelled by Stern) have a dipolar structure with zero net angular momentum; general solutions, including the axisymmetric rider, must also have a vanishing angular momentum in order to be stationary, i.e., the azimuthal velocity of the rider must change sign so that it is not a monopole [3]. However, zonally propagating structures on the β-plane will not lead to any meridional transport.

The variation of the intensity ratio of partners and corresponding change in the path curvature of non-zonally propagating dipoles have been described approximately by the well-known equation of the physical pendulum [10]. The numerical investigation of the dynamics of equivalent-barotropic f-plane dipoles which are steady solutions in the absence of the β-effect, demonstrated that they remain coherent on a β-plane if dipoles are strong enough [11]. Thus, the β-effect is not crucial for an intense dipole when the swirling velocity in the partners are much higher than the Rossby wave speed as observed in the upper ocean. The curved path of the vortex couple resulting from different strengths of companions may originate from the formation process and be

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affected only slightly by the β-effect as demonstrated in laboratory experiments [7].

The vortex evolution can be also strongly influenced by nonuniform background currents. However, the evolution of vortex dipoles in external flows have received only little attention in literature. In particular, a horizontal strain would either accelerate the dipole and form a head-tail structure, or separate the partners, depending on the strain orientation [12]. Under the influence of a radial flow from a point source (or sink), dipoles can separate, converge or follow spiraling trajectories [13].

In the present study, we consider evolution of vortex couples in a horizontally sheared nondivergent flow with constant potential vorticity. Vortex couples are considered in the point vortex approximation that allows to express the solution in quadratures for the case of axisymmetric external flow and analyze different dynamical regimes explicitly for an external flow typical for a topographic circulation around an isolated seamount [14]. The mathematical formulation is presented in Section 2. The solution in the general form for arbitrary ratio of point vortex intensities is described in Section 3. The behavior of a dipolar couple approaching from the infinity is analyzed in Section 4. Discussion and conclusions are in Section 5.

2. Mathematical formulation

2.1. Potential vorticity equation

At the lowest order of approximation, quasi-two-dimensional dynamics of oceans, planetary atmospheres and of plasmas in a magnetic field are governed by the conservation of a material invariant Q in an equivalent-barotropic model [5]

$$\partial_t Q + \mathbf{u}\nabla Q = 0. \tag{1}$$

Here t is time normalized by a time scale T, (x, y) are horizontal coordinates normalized by a horizontal scale L, $\mathbf{u}$ is the flow velocity normalized by L/T. In geophysical fluid applications such model describes a thin layer of homogeneous, incompressible fluid, strongly constrained by ambient rotation and by stratification, overlying an infinitely deep layer of fluid at rest [6]. Then potential vorticity Q represents the ratio of absolute vorticity to the layer thickness and it is conserved in each fluid parcel.

For small Rossby number and order unity Burger number, the velocity is nondivergent in the leading order, so that the velocity and potential vorticity are expressed by the streamfunction, $\psi(x, y, t)$ normalized by L^2/T

$$\mathbf{u}(x, y, t) = \mathbf{k}\times\nabla\psi, \qquad Q = \nabla^2\psi - S, \tag{2}$$

where $\mathbf{k}$ is the vertical unit vector, and the term S is the vortex stretching related to changes of the layer thickness either due to localized topography $S(x, y)$, or due to geostrophic adjustment outside topography, where $S = \gamma^2\Psi$. Here $\gamma = fL/C_g$, the gravity wave speed, C_g, and the Coriolis parameter, f, are assumed to be constants. For the plasma case, ψ is proportional to the electrostatic potential normalized by T_e/e (T_e is the electron temperature, e is the electric charge of the ion), $C_g = \sqrt{T_e/m_i}$ is the ion sound speed (m_i is the ion mass), and $f = \omega_{ic}$ is the ion cyclotron frequency [15].

In the limiting case of $\gamma \to 0$, (1)–(2) describe pure two-dimensional incompressible flows and they are valid for an arbitrary Rossby number. In both models, the domain is infinite in horizontal directions.

2.2. Point vortex model

Following [16], we decompose the flow into a steady external part, $\mathbf{U} = \mathbf{k}\times\nabla\Psi$, and a vortical part $\mathbf{u}_V = \mathbf{u} - \mathbf{U}$ which corresponds to localized vortices.

Assuming that the vortex dipole is represented by localized potential vorticity anomalies with amplitudes κ_1 and κ_2, their self-induced drift affected by the external flow is governed by the following equations

$$\frac{dx_1}{dt} = -\partial_y\Psi(x_1, y_1) + \kappa_2(y_2 - y_1)\omega(r_{12}),$$
$$\frac{dy_1}{dt} = \partial_x\Psi(x_1, y_1) - \kappa_2(x_2 - x_1)\omega(r_{12}), \tag{3}$$
$$\frac{dx_2}{dt} = -\partial_y\Psi(x_2, y_2) - \kappa_1(y_2 - y_1)\omega(r_{12}),$$
$$\frac{dy_2}{dt} = \partial_x\Psi(x_2, y_2) + \kappa_1(x_2 - x_1)\omega(r_{12}), \tag{4}$$

where $r_i^2 = x_i^2 + y_i^2$, $r_{12}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$, and $\omega(r)$ is the rotation rate of a point vortex, either in the equivalent-barotropic model, or in the 2D model:

$$\omega(r) = \frac{d\psi}{r\,dr}, \qquad \psi = -\frac{1}{2\pi}K_0(\gamma r), \quad \text{or} \quad \psi = \frac{1}{2\pi}\ln r. \tag{5}$$

The system of ODE (3)–(4) conserves the Hamiltonian $\mathcal{H}$ [2]:

$$\mathcal{H} = \Psi(r_1) + q\Psi(r_2) + q\kappa_1\psi(r_{12}) = \text{const}, \tag{6}$$

where $q = \kappa_2/\kappa_1$ while the components of linear momentum and the total angular momentum, M, evolve due to the external flow:

$$\dot{x}_1 + q\dot{x}_2 = -\partial_y\Psi(x_1, y_1) - q\partial_y\Psi(x_2, y_2),$$
$$\dot{y}_1 + q\dot{y}_2 = \partial_x\Psi(x_1, y_1) + q\partial_x\Psi(x_2, y_2), \tag{7}$$
$$\dot{M} = (y_1\partial_x - x_1\partial_y)\Psi(x_1, y_1) + q(y_2\partial_x - x_2\partial_y)\Psi(x_2, y_2),$$
$$2M = r_1^2 + qr_2^2. \tag{8}$$

Therefore, analytical solutions for point vortex evolution in an external flow were found only in cases with additional symmetry considered in [13].

As one can see from (8), the total angular momentum is conserved $M = \text{const}$ when the external flow is axisymmetric and stationary on the f-plane,

$$U = -y\Omega(r), \qquad V = x\Omega(r), \qquad \Omega = \frac{d\Psi}{r\,dr}, \tag{9}$$

where Ω is the external rotation rate and $r^2 = x^2 + y^2$.

Conservation of total angular momentum in this case, added to the Hamiltonian conservation, allows a reduction from the original four degrees of freedoms to only two and a general solution in quadratures.

3. Motion of a vortex dipole in the general case

3.1. Simple case of constant external rotation rate

When $\Omega = \text{const}$, the solution is obvious in coordinates rotating with the angular rate Ω: the distance between partners is known to remain constant $r_{12} = \text{const}$, while the center of the dipole defined as $x_c = (x_1 + x_2)/2$, $y_c = (y_1 + y_2)/2$ moves along circular trajectories with radius $r_q = r_{12}(1 - q)/[2(1 + q)]$. In particular, equal vortices of the same sign ($q = 1$) rotate around each other ($r_q = 0$), while a dipole with opposite sign vortices ($q = -1$) propagates uniformly ($r_q = \infty$) with the speed $U_\infty = r_{12}\omega(r_{12})$.