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Higher accuracy analytical approximations to a nonlinear oscillator with discontinuity by He's homotopy perturbation method

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Abstract

He's homotopy perturbation method is used to calculate higher-order approximate periodic solutions of a nonlinear oscillator with discontinuity for which the elastic force term is proportional to $\text{sgn}(x)$. We find He's homotopy perturbation method works very well for the whole range of initial amplitudes, and the excellent agreement of the approximate frequencies and periodic solutions with the exact ones has been demonstrated and discussed. Only one iteration leads to high accuracy of the solutions with a maximal relative error for the approximate period of less than 1.56% for all values of oscillation amplitude, while this relative error is 0.30% for the second iteration and as low as 0.057% when the third-order approximation is considered. Comparison of the result obtained using this method with those obtained by different harmonic balance methods reveals that He's homotopy perturbation method is very effective and convenient.

Keywords: Nonlinear oscillator; Approximate solutions; Homotopy perturbation method

1. Introduction

The study of nonlinear problems is of crucial importance not only in all areas of physics but also in engineering and other disciplines, since most phenomena in our world are essentially nonlinear and are described by nonlinear equations. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear problem. There are several methods used to find approximate solutions to nonlinear problems, such as perturbation techniques [1–6], harmonic balance based methods [6–10] or other techniques [11–17]. An excellent review on some asymptotic methods for strongly nonlinear equations can be found in detail in Refs. [18] and [19].

In the present Letter we obtain higher-order analytical approximations to the periodic solutions to a nonlinear oscillator with discontinuity for which the elastic restoring force is an antisymmetric and constant force. To do this, we apply He's homotopy perturbation method [20–38]. This type of oscillator has been analyzed by Liu [39] and by Özis and Yildirim [34] applying the first-order homotopy perturbation method. They also make the comparison of two distinct adaptations of the first-order homotopy perturbation method for determining frequency-amplitude relation of the nonlinear oscillator with discontinuities. However, higher-order analytical approximate solutions have not been obtained for this oscillator using He's homotopy perturbation method. As we can see, the results presented in this Letter reveal that the method is very effective and convenient for conservative nonlinear oscillators for which the restoring force has a non-polynomial form.

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2. Solution procedure

This Letter considers the following nonlinear oscillator with discontinuity

$$\frac{d^2x}{dt^2} + \mathrm{sgn}(x) = 0 \tag{1}$$

with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{dx}{dt}(0) = 0 \tag{2}$$

and $\mathrm{sgn}(x)$ is defined by

$$\mathrm{sgn}(x) = \begin{cases} -1, & x < 0, \\ +1, & x \geqslant 0. \end{cases} \tag{3}$$

All the solutions to Eq. (1) are periodic. We denote the angular frequency of these oscillations by ω and note that one of our major tasks is to determine $\omega(A)$, i.e., the functional behaviour of ω as a function of the initial amplitude A.

There is no small parameter in Eq. (1), so the traditional perturbation methods cannot be applied directly, moreover the equation involves discontinuity. Due to the fact that the homotopy perturbation method requires neither a small parameter nor a linear term in a differential equation, we can approximately solve Eq. (1) using the homotopy perturbation method. In this method, an artificial perturbation equation is constructing by embedding an artificial parameter $p \in [0, 1]$, which is used as expanding parameter. This technique yields a very rapid convergence of the solution series; in most cases, only one iteration leads to high accuracy of the solution. This method provides an effective and convenient mathematical tool for nonlinear differential equations [18,19].

Eq. (1) can be re-written in the form

$$\frac{d^2x}{dt^2} + x = x - \mathrm{sgn}(x). \tag{4}$$

For Eq. (4) we can establish the following homotopy

$$\frac{d^2x}{dt^2} + x = p\big[x - \mathrm{sgn}(x)\big], \tag{5}$$

where p is the homotopy parameter. When $p = 0$, Eq. (5) becomes a linear differential equation for which an exact solution can be calculated; for $p = 1$, Eq. (5) then becomes the original problem. Now the homotopy parameter p is used to expand the solution $x(t)$ and the square of the unknown angular frequency ω as follows

$$x(t) = x_0(t) + px_1(t) + p^2x_2(t) + \cdots, \tag{6}$$

$$1 = \omega^2 - p\alpha_1 - p^2\alpha_2 - \cdots, \tag{7}$$

where α_i $(i = 1, 2, \ldots)$ are to be determined.

Substituting Eqs. (6) and (7) into Eq. (5)

$$\begin{aligned} &\big(x_0'' + px_1'' + p^2x_2'' + \cdots\big) + \big(\omega^2 - p\alpha_1 - p^2\alpha_2 - \cdots\big)\big(x_0 + px_1 + p^2x_2 + \cdots\big) \\ &\quad = p\big[\big(x_0 + px_1 + p^2x_2 + \cdots\big) - \mathrm{sgn}\big(x_0 + px_1 + p^2x_2 + \cdots\big)\big] \end{aligned} \tag{8}$$

and collecting the terms of the same power of p, we obtain a series of linear equations, of which we write only the first four

$$x_0'' + \omega^2x_0 = 0, \quad x_0(0) = A, \quad x_0'(0) = 0, \tag{9}$$

$$x_1'' + \omega^2x_1 = (1 + \alpha_1)x_0 - \mathrm{sgn}(x_0), \quad x_1(0) = x_1'(0) = 0, \tag{10}$$

$$x_2'' + \omega^2x_2 = \alpha_2x_0 + (1 + \alpha_1)x_1, \quad x_2(0) = x_2'(0) = 0, \tag{11}$$

$$x_3'' + \omega^2x_3 = \alpha_3x_0 + \alpha_2x_1 + (1 + \alpha_1)x_2, \quad x_3(0) = x_3'(0) = 0. \tag{12}$$

In Eqs. (9)–(12) we have taken into account the following expression

$$\begin{aligned} f(x) &= f\big(x_0 + px_1 + p^2x_2 + p^3x_3 + \cdots\big) \\ &= f(x_0) + p\left(\frac{df(x)}{dx}\right)_{x=x_0} x_1 + p^2\left[\left(\frac{df(x)}{dx}\right)_{x=x_0} x_2 + \frac{1}{2}\left(\frac{d^2f(x)}{dx^2}\right)_{x=x_0} x_1^2\right] + O\big(p^3\big), \end{aligned} \tag{13}$$

where $f(x) = \mathrm{sgn}(x)$ and

$$\frac{d\,\mathrm{sgn}(x)}{dx} = \frac{d^2\,\mathrm{sgn}(x)}{dx^2} = \cdots = 0 \quad \text{for } x \neq 0,$$