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Chaos control via a simple fractional-order controller

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Abstract

In this Letter, we propose a fractional-order controller to stabilize the unstable fixed points of an unstable open-loop system. Also, we show that this controller has strong ability to eliminate chaotic oscillations or reduce them to regular oscillations in the chaotic systems. This controller has simple structure and is designed very easily. To determine the control parameters, one needs only a little knowledge about the plant and therefore, the proposed controller is a suitable choice in the control of uncertain chaotic systems.

Keywords: Fractional-order controller; Chaos control

1. Introduction

Chaos theory, as a new branch of physics and mathematics, has provided us a new way of viewing the universe and is an important tool to understand the world we live in. Chaotic behaviors have been observed in different areas of science and engineering such as mechanics, electronics, physics, medicine, ecology, biology, economy and so on. To avoid troubles arising from unusual behaviors of a chaotic system, chaos control has gained increasing attention in recent years. An important objective of a chaos controller is to suppress the chaotic oscillations completely or reduce them to the regular oscillations [1]. Many control techniques such as open-loop control methods, traditional linear and nonlinear control methods, adaptive control methods, and fuzzy control methods have heretofore been implemented in the control of chaotic systems [1,2].

Fractional calculus is a mathematical topic with more than 300 years old history but its application to physics and engineering has been attracted lots of attention only in the recent years. It has been found that in interdisciplinary fields, many systems can be described by fractional differential equations. For example dielectric polarization [3], electrode–electrolyte polarization [4], electromagnetic waves [5], visco-elastic systems [6], quantum evolution of complex systems [7], quantitative finance [8] and diffusion wave [9] have been known to display fractional-order dynamics. Due to the lack of appropriate mathematical methods [10], fractional-order dynamic systems were studied only marginally in the design and practice of control systems in the last few decades. However, in the recent years, emergence of effective methods in differentiation and integration of non-integer order equations makes fractional-order systems more and more attractive for the systems control community. The TID controller [11], the $PI^{\lambda}D^{\mu}$ controller [10], the CRONE controllers [12–14] and the fractional lead-lag compensator [15,16] are some of the well-known fractional-order controllers. In some of these papers it is verified that the fractional-order controllers can have better disturbance rejection ratios and less sensitivity to plant parameter variations compared to the traditional controllers.

In this Letter, we propose a simple fractional-order controller to control unstable systems. Also, we show that the proposed controller can reduce chaotic oscillations to the regular oscillations or eliminate them. This Letter is organized as follows. Section 2 includes basic concepts in fractional calculus. In Section 3, a fractional-order controller is proposed to stabilize unstable fixed points of an open-loop system. Sec-

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tion 4 discusses the chaos control issue. In this section, the proposed fractional-order controller is employed to control some chaotic systems. Conclusions in Section 5 close the Letter.

2. Introduction to fractional calculus

2.1. Definitions

The differintegral operator, denoted by ${}_aD_t^q$, is a combined differentiation–integration operator commonly used in fractional calculus. This operator is a notation for taking both the fractional derivative and the fractional integral in a single expression. For positive q it denotes derivative and for negative q it denotes integral actions.

The commonly used definitions for fractional derivatives are Grunwald–Letnikov, Riemann–Liouville and Caputo definitions [17]. The Grunwald–Letnikov definition is given by:

$$
\begin{aligned}
&{}_aD_t^q f(t)\\
&\quad=\frac{d^q f(t)}{d(t-a)^q}\\
&\quad=\lim_{N\to\infty}\left[\frac{t-a}{N}\right]^{-q}\sum_{j=0}^{N-1}(-1)^j\binom{q}{j}f\left(t-j\left[\frac{t-a}{N}\right]\right). \qquad (1)
\end{aligned}
$$

The Riemann–Liouville definition is the simplest and easiest definition to use. This definition is given by:

$$
{}_aD_t^q f(t) := \begin{cases} \frac{1}{\Gamma(-q)}\int_a^t (t-\tau)^{-q-1}f(\tau)\,d\tau, & q<0,\\ f(t), & q=0,\\ D^n[{}_aD_t^{q-n}f(t)], & q>0, \end{cases} \qquad (2)
$$

where n is the first integer larger than q, i.e., $n-1 \leqslant q < n$ and Γ is the Gamma function:

$$
\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt. \qquad (3)
$$

The Laplace transforms of the Riemann–Liouville fractional integral and derivative are given as follows:

$$
L\{{}_0D_t^q f(t)\} = s^q F(s), \quad q \leqslant 0, \qquad (4)
$$

$$
\begin{aligned}
&L\{{}_0D_t^q f(t)\} = s^q F(s) - \sum_{k=0}^{n-1} s^k\,{}_0D_t^{q-k-1}f(0),\\
&\quad n-1<q\leqslant n\in N. \qquad (5)
\end{aligned}
$$

Unfortunately, the Riemann–Liouville fractional derivative appears unsuitable to be treated by the Laplace transform technique in that it requires knowledge of the non-integer order derivatives of the function at $t=0$. This trouble does not exist in the Caputo definition of the fractional derivative. This definition of derivative, which sometimes called smooth fractional derivative, is defined by:

$$
{}_0D_t^q f(t) = \begin{cases} \frac{1}{\Gamma(m-q)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}}\,d\tau, & m-1<q<m,\\ \frac{d^m}{dt^m}f(t), & q=m, \end{cases} \qquad (6)
$$

where m is the first integer larger than q. The Laplace transform of the Caputo fractional derivative is:

$$
\begin{aligned}
&L\{{}_0D_t^q f(t)\} = s^q F(s) - \sum_{k=0}^{n-1} s^{q-1-k} f^{(k)}(0),\\
&\quad n-1<q\leqslant n\in N. \qquad (7)
\end{aligned}
$$

Contrary to the Laplace transform of the Riemann–Liouville fractional derivative, only integer order derivatives of function f are appeared in the Laplace transform of the Caputo fractional derivative. For zero initial conditions, (7) reduces to:

$$
L\{{}_0D_t^q f(t)\} = s^q F(s). \qquad (8)
$$

2.2. Approximations

Direct implementation of fractional-order transfer functions is problematic. Hence, to implement these transfer functions, integer order approximations of the fractional transfer functions are determined. There are many different methods to find such approximations. Charef [18,19], Oustaloup [20], Carlson [21] and Matsuda [22] approximations are the well-known approximations of the fractional-order transfer functions. From the control theoretic point of view, the proposed methods are divided into two groups. Methods that use continued fraction expansions (CFE) and interpolation techniques such as Carlson and Matsuda methods, and methods that use curve fitting or identification techniques such as Oustaloup and Charef methods [23].

Since none of the existing methods transcend others regarding to all desires, it is not possible to say that which one is the best [24]. A comprehensive comparison of these approximation methods has been given in Chapter 3 of [24]. In the numerical simulations of the present work, the Oustaloup method is used to find rational approximation of the fractional operators. This simple method provides a continuous approximation of fractional transfer functions using recursive allocation of zeros and poles to achieve an admissible accuracy.

The approximated transfer function based on Oustaloup method is determined using the following definition:

$$
s^\nu \approx k\prod_{n=1}^{N}\frac{1+s/\omega_{z,n}}{1+s/\omega_{p,n}}, \quad 0<\nu<1. \qquad (9)
$$

Gain k is adjusted so that both sides of (9) have unit gain at 1 rad/s. The number of poles and zeros of the approximated transfer function (N) and the frequency range ($[\omega_l,\omega_h]$) are selected beforehand. $\omega_{z,n}$ and $\omega_{p,n}$ are calculated by the following equations:

$$
\begin{aligned}
&\omega_{z,1}=\omega_l\sqrt{\eta},\\
&\omega_{p,n}=\omega_{z,n}\xi, \quad n=1,\ldots,N,\\
&\omega_{z,n}=\omega_{p,n-1}\eta, \quad n=2,\ldots,N, \qquad (10)
\end{aligned}
$$