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The width of a chaotic layer

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Abstract

A model of nonlinear resonance as a periodically perturbed pendulum is considered, and a new method of analytical estimating the width of a chaotic layer near the separatrices of the resonance is derived for the case of slow perturbation (the case of adiabatic chaos). The method turns out to be successful not only in the case of adiabatic chaos, but in the case of intermediate perturbation frequencies as well.

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1. Introduction

The extent of chaotic domains, and, in particular, the width of chaotic layers, is one of the most important characteristics of the chaotic motion of Hamiltonian systems. Until now, several aspects of the problem of analytical estimation of the width of a chaotic layer were considered in Refs. [1–7]. Potentially, the ability of estimating the extent of chaos in phase space of Hamiltonian systems has a wide field of applications in physics and dynamical astronomy. Wisdom et al. [8] and Wisdom [9] estimated the width of the chaotic layer near the separatrices of spin–orbit resonances in the rotational dynamics of planetary satellites and Mercury. Yamagishi [10] made estimates of the width of the chaotic layer near the magnetic separatrix in poloidal diverter tokamaks. In these both applications, Chirikov's approach [2] based on the separatrix map theory was used. Chirikov derived approximate formulas for the width in the assumption of high-frequency perturbation of nonlinear resonance; however, as follows from these same formulas, the chaotic layer is exponentially thin with the ratio of perturbation frequency to the frequency of small-amplitude phase oscillations on the resonance. This means that the cases of intermediate and low frequencies of perturbation are most actual in applications. So, analysis of the problem of estimation of the width of a chaotic layer in these cases is definitely necessary.

In this Letter, a method of analytical estimation of the width of a chaotic layer, especially aimed at the case of slow, or adiabatic, chaos, is proposed. It is based on the theory of separatrix maps. The nonlinear resonance is modelled by the Hamiltonian of a perturbed nonlinear pendulum. There are two fundamental parameters: the ratio of the frequency of perturbation to the frequency of small-amplitude phase oscillations on the resonance, and the parameter characterizing strength of the perturbation.

The applicability of the theory of separatrix maps for description of the motion near the separatrices of the perturbed nonlinear resonance in the full range of the relative frequency of perturbation, including its low values, was discussed and shown to be legitimate in Ref. [11].

The field of applications of the derived method is rather wide due to generic character [2] of the perturbed pendulum model of nonlinear resonance. The method can be used in any application where a separatrix map is derived for description of chaotic motion. Many of such applications are described, e.g., in Ref. [12].

Analytical and numerical approaches to measuring the width of a chaotic layer have different merits and different demerits. The inherent shortcoming of any analytical approach consists in that it implies an idealization of the phenomenon, and the estimates are inherently approximate. The precision of the estimates is hard to evaluate, due to a number of approximations involved. On the other hand, the numerical methods are applica-

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ble in a rather narrow range of values of parameters: they cannot be used in the case of very low relative frequencies of perturbation (due to limitations on computation time), also in the case of high relative frequencies of perturbation (because the width of the chaotic layer is exponentially thin with the perturbation frequency), and in the case of very small amplitudes of perturbation, due to limitations on the arithmetic precision. Therefore only analytical methods can give the global picture. Their another advantage is that the analytical estimation is easy and fast, as soon as the theoretical model is shown to be valid. Finally, the most important advantage, perhaps, is in the physical insight that the analytical methods provide, making the role of each parameter clearly visible.

2. The model of nonlinear resonance and the separatrix map

Under general conditions [2,13,14], a model of nonlinear resonance is provided by the Hamiltonian of the nonlinear pendulum with periodic perturbations. A number of problems on nonlinear resonances in mechanics and physics is described by the Hamiltonian

$$H = \frac{\mathcal{G}p^2}{2} - \mathcal{F}\cos\varphi + a\cos(k\varphi - \tau) + b\cos(k\varphi + \tau) \qquad (1)$$

(see, e.g., Ref. [11]). The first two terms in Eq. (1) represent the Hamiltonian H_0 of the unperturbed pendulum; φ is the pendulum angle (the resonance phase angle), p is the momentum. The periodic perturbations are given by the last two terms; τ is the phase angle of perturbation: $\tau = \Omega t + \tau_0$, where Ω is the perturbation frequency, and τ_0 is the initial phase of the perturbation. The quantities $\mathcal{F}$, $\mathcal{G}$, a, b, k are constants. We assume that $\mathcal{F} > 0$, $\mathcal{G} > 0$, k is integer, and $a = b$. We use the notation $\varepsilon \equiv a/\mathcal{F} = b/\mathcal{F}$ for the relative amplitude of perturbation.

The so-called separatrix (or "whisker") map

$$\begin{aligned} w_{i+1} &= w_i - W\sin\tau_i, \\ \tau_{i+1} &= \tau_i + \lambda\ln\frac{32}{|w_{i+1}|} \pmod{2\pi}, \end{aligned} \qquad (2)$$

written in the present form and explored in Refs. [1,2,13] and first introduced in Ref. [15], describes the motion in the vicinity of the separatrices of Hamiltonian (1). The quantity w denotes the relative (with respect to the unperturbed separatrix value) pendulum energy $w \equiv \frac{H_0}{\mathcal{F}} - 1$, and τ retains its meaning of the phase angle of perturbation. The constants λ and W are the two basic parameters, already mentioned in the Introduction. The parameter λ is the ratio of Ω, the perturbation frequency, to $\omega_0 = (\mathcal{F}\mathcal{G})^{1/2}$, the frequency of the small-amplitude pendulum oscillations. The parameter W in the case of $k = 1$ and $a = b$ has the form [16]:

$$W = \varepsilon\lambda\bigl(A_2(\lambda) + A_2(-\lambda)\bigr) = \frac{4\pi\varepsilon\lambda^2}{\sinh\frac{\pi\lambda}{2}}. \qquad (3)$$

Here $A_2(\lambda) = 4\pi\lambda\frac{\exp(\pi\lambda/2)}{\sinh(\pi\lambda)}$ is the value of the Melnikov–Arnold integral as defined in Ref. [2]. Formula (3) differs from that given in Refs. [2,14] by the term $A_2(-\lambda)$, which is small for $\lambda \gg 1$. However, its contribution is significant for λ small [16], i.e., in the case of adiabatic chaos. Expression (3) for the parameter W needs to be modified at very high relative frequencies of perturbation (see Refs. [5,17]). Analytical expressions for W at different values of k are given in Refs. [2,11] and at arbitrary a, b in Ref. [11].

The accuracy of separatrix map (2) in describing the behaviour of original system (1) can be estimated by the order of magnitude as $\sim \varepsilon$ (see Refs. [5,12]). Measurement of the chaotic layer width allows one to estimate the accuracy directly, as demonstrated below in Section 5.

Note that the expression for the increment of the phase τ in map (2) is a rough approximation. It is valid for a low strength of perturbation, i.e., at $w \ll 1$. According to Refs. [16,18], one can improve the accuracy of the map by means of replacing the logarithmic approximation of the phase increment by the original expression through the elliptic integrals. For the sake of brevity we do not explore the advantages of this improvement further in estimating the width. This can be straightforwardly accomplished if one needs to improve precision of estimating the width at increasing the magnitude of perturbation.

One iteration of map (2) corresponds to one period of the pendulum rotation or a half-period of its libration. The motion of system (1) is mapped by Eq. (2) asynchronously [16]: the relative energy variable w is taken at $\varphi = \pm\pi$, while the perturbation phase τ is taken at $\varphi = 0$. The desynchronization can be removed by a special procedure [11,16]. The synchronized separatrix map gives correct representation of the sections of the phase space of the near-separatrix motion both at high and low perturbation frequencies; this was found in Ref. [11] by direct comparison of phase portraits of the separatrix map to the corresponding sections obtained by numerical integration of the original systems. This testifies good performance of both the separatrix map theory and the Melnikov theory (that describes the splitting of the separatrices).

The asymptotic expression for W that ensues from Eq. (3) at $\lambda \sim 0$ is $W \approx 8\varepsilon\lambda$. A good correspondence of this expression to the actual amplitude of the separatrix map derived numerically by integration of the original system was found in Ref. [6].

An equivalent form of Eq. (2), used, e.g., in Refs. [16,19], is

$$\begin{aligned} y_{i+1} &= y_i + \sin x_i, \\ x_{i+1} &= x_i - \lambda\ln|y_{i+1}| + c \pmod{2\pi}, \end{aligned} \qquad (4)$$

where $y = w/W$, $x = \tau + \pi$; and

$$c = \lambda\ln\frac{32}{|W|}. \qquad (5)$$

Note that while the second line of Eqs. (4) is taken modulo 2π, the quantity c given by Eq. (5) is taken modulo 2π in the following analytical treatment only when explicitly stated.

3. The case of the least perturbed border

We obtain the dependence of the half-width y_b of the main chaotic layer of the separatrix map in a numerical experiment with Eqs. (4). The border value y_b, corresponding to