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Physical dynamics of quasi-particles in nonlinear wave equations

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Abstract

By treating the centers of solitons as point particles and studying their discrete dynamics, we demonstrate a new approach to the quantization of the soliton solutions of the sine-Gordon equation, one of the first model nonlinear field equations. In particular, we show that a linear superposition of the non-interacting shapes of two solitons offers a qualitative (and to a good approximation quantitative) description of the true two-soliton solution, provided that the trajectories of the centers of the superimposed solitons are considered *unknown*. Via variational calculus, we establish that the dynamics of the quasi-particles obey a pseudo-Newtonian law, which includes cross-mass terms. The successful identification of the governing equations of the (discrete) quasi-particles from the (continuous) field equation shows that the proposed approach provides a basis for the passage from the continuous to a discrete description of the field.

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1. Introduction

A sought-after property of model field equations is that they possess localized, permanent wave solutions that retain their identity upon interacting with each other. *Solitons*, which are solutions of *fully-integrable* equations, are an example of such waves. Unfortunately, integrability is not always a property of models that are of physical importance. Therefore, it is important to develop simple, albeit approximate, approaches to studying the dynamics of permanent waves in non-integrable systems, for which exact solutions are difficult to obtain.

The idea of identifying a localized solution of a nonlinear wave equation with an elementary particle was first proposed by Perring and Skyrme [1]. They found a solution of (what is known today as) the sine-Gordon equation (SGE) [2] consisting of two interacting, localized waves, which is an example of (what is nowadays called) a *two-soliton solution*, several years before the notion of a soliton was introduced by Zabusky and Kruskal [3]. Through numerical simulation, the latter authors discovered that the localized traveling-wave solutions of the Korteweg–de Vries equation retain their shapes (identities) after they pass through each other (interact). Apparently, Zabusky and Kruskal were unaware of the work of Perring and Skyrme [1] and arrived at the idea of identifying the nonlinear waves' dynamics with those of particles through experiment rather than conjecture.

Since their discovery, solitons have attracted an enormous amount of attention. Significant progress has been made in their mathematical description, and their applications have been far-reaching [4,5]. The so-called kink solitary waves considered by Perring and Skyrme [1] have been shown to be indeed solitons [5,6]. Moreover, the relationship between the particle-like dynamics of the coherent structures that emerge in the solutions of nonlinear wave equations and the field theories of particle physics are well-established in the literature [4,7–9].

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From a mathematical point of view, we can elucidate the latter relationship by, somehow, reducing the "infinitely complex" continuous description of the field to a "finitely complex" discrete description. In this Letter, we show how this can be achieved by studying the dynamics of *quasi-particles*. Our approach amounts to "degrading" the continuous description of a wave profile to a discrete description of the centers of coherent structures, assuming that the shapes of the coherent structures (for an integrable system, this would simply be the solitons) are not significantly affected by each other's presence. To this end, in this Letter, the term 'quasi-particle' refers to these permanent, indestructible and virtually non-deformable coherent structures, whose centers can be treated as point particles with mass equal to some associated measure of inertia that we call the *pseudomass*. Furthermore, replacing a complicated continuous profile by a superposition of localized shapes can also be viewed as a *coarse-grain description*, in the sense that small deformations and wiggles are filtered out from the profile leaving merely the main structure (the "grains").

Hence, the coarse-grain description amounts to replacing the solution of a nonlinear wave equation with a linear superposition of basis states (i.e., traveling-wave solutions in their undeformed, or non-interacting, state), whose centers' trajectories, which for a nonlinear equation do not follow the undisturbed (linear) trajectories, are considered unknown. Restricting to the case of just two superimposed waves, a discrete model for the trajectories is derived and solved numerically in this Letter. This approach gives a wave profile whose deviation from the analytical two-soliton solution (when available) is orders of magnitude smaller than the characteristic height of the solitons. Consequently, this paves the way for the construction of successive approximations that account for the higher-order, nonlinear interactions of solitons.

Here, we note that our coarse-grain description is a special interpretation of the more general method of *collective coordinates/variables*, which has been put on solid theoretical ground [10] and has become part of the textbooks on solitons [5]. The latter approach made its debut in the study of resonances and collisions of solitary waves in the so-called ϕ^4 equation [11,12], a close relative of the SGE, and the study of two-soliton interactions in the SGE [13,14]. Moreover, the method of collective variables is just one type of *variational approximation*, which is another approach to the analytical study of nonlinear wave equations that has recently regained popularity [15–17]. Furthermore, it appears that Rice [18] was the first to realize that the collective-coordinate variational approximation provides a way of "distilling" the particle-like dynamics of nonlinear waves from the continuous (field) description, though Karpman and Solov'ev [13] had the foresight to use the term 'quasi-particle' in their discussion.

Finally, we note that the nonlinear wave equation featured herein—the sine-Gordon equation—continues to be of interest as a model field equation [19]. In addition, various modifications of it have been considered in the literature. For example, in order to establish the effects of acceleration on the shape of the SGE's solitons, Fogel et al. [20] introduced a driving force into the SGE. However, this required also adding dissipation in the field equation in order to stabilize the evolution of the solitons, i.e., to ensure that they reach a steady terminal velocity [20,21]. Adding dissipation opens new horizons of investigation, and different physical mechanisms can be considered as progenitors of the dissipative force. It is well known that, in fluid mechanics, linear dissipation of either viscous or Darcy type can balance the nonlinearity in the field equation and allow stable localized waves to exist [22,23]. Similarly, in incompressible shallow-water flows, a viscous dissipation can allow for the existence of localized coherent structures with solitonic behavior [24]. Nonetheless, one thing is certain: dissipation can alter the behavior of a nonlinear field equation dramatically. Therefore, in this Letter, we focus on the *lossless* SGE of Perring and Skyrme [1] and show that the coarse-grain description of the field leads to the physical dynamics of *locally-accelerating* quasi-particles, *without* introducing a driving force or dissipation into the equation. In this respect, however, the SGE differs fundamentally from the modern lossless nonlinear field theories of continuum mechanics (see, e.g., Ref. [25] and those therein), since the "unbalanced" nonlinearity in the former allows for the creation of localized coherent structures and does not lead to formation of singularities in finite time.

2. The sine-Gordon equation and its soliton solutions

For the purposes of this Letter, the SGE takes the following dimensionless ($\hbar = c = m = 1$) form:

$$u_{tt} - u_{xx} = -\sin u, \tag{1}$$

where the subscripts denote partial differentiation. We have selected the SGE as our featuring example because there are known analytical expressions for its two-soliton solutions. This allows us to show that the coarse-grain description is both an effective approximation tool and a new method for nonlinear-wave quantization.

It is easy to show that the Lagrangian and the Hamiltonian of the SGE read

$$L, H = \int_{-\infty}^{+\infty} \frac{1}{2}u_t^2 \mp \left(\frac{1}{2}u_x^2 + 1 - \cos u\right) \mathrm{d}x, \tag{2}$$

respectively [2,5], where the "−" sign in the right-hand side refers to L and the "+" sign to H. In addition, the *wave momentum* is defined [7] as

$$P = -\int_{-\infty}^{+\infty} u_x u_t \, \mathrm{d}x. \tag{3}$$

Then, the conservation of the energy and linear momentum require that $\mathrm{d}H/\mathrm{d}t = 0$ and $\mathrm{d}P/\mathrm{d}t = 0$, respectively.

Now, if we consider the moving frame $\xi = x - vt$, Eq. (1) reduces to an ODE, which has the following solution:

$$u = \phi(\xi) = 4\arctan\left[\exp\left(\frac{\xi}{\sqrt{1-v^2}}\right)\right], \quad 0 \leqslant v < 1. \tag{4}$$

Notice that the latter is a *one-soliton solution* because $\frac{1}{2\pi}[\lim_{x\to+\infty} u(x,t) - \lim_{x\to-\infty} u(x,t)] = 1$ for all $t < \infty$ [2].