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Harmonic balance approach to the periodic solutions of the (an)harmonic relativistic oscillator

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Abstract

The first-order harmonic balance method via the first Fourier coefficient is used to construct two approximate frequency–amplitude relations for the relativistic oscillator for which the nonlinearity (anharmonicity) is a relativistic effect due to the time line dilation along the world line. Making a change of variable, a new nonlinear differential equation is obtained and two procedures are used to approximately solve this differential equation. In the first the differential equation is rewritten in a form that does not contain a square-root expression, while in the second the differential equation is solved directly. The approximate frequency obtained using the second procedure is more accurate than the frequency obtained with the first due to the fact that, in the second procedure, application of the harmonic balance method produces an infinite set of harmonics, while in the first procedure only two harmonics are produced. Both approximate frequencies are valid for the complete range of oscillation amplitudes, and excellent agreement of the approximate frequencies with the exact one are demonstrated and discussed. The discrepancy between the first-order approximate frequency obtained by means of the second procedure and the exact frequency never exceeds 1.6%. We also obtained the approximate frequency by applying the second-order harmonic balance method and in this case the relative error is as low 0.31% for all the range of values of amplitude of oscillation A.

Keywords: Nonlinear oscillations; Relativistic oscillator; Harmonic balance method; Approximate frequency

1. Introduction

The study of nonlinear oscillators is of great interest in engineering and physical sciences and many analytical techniques have been developed for solving the second-order nonlinear differential equations that govern their motion [1]. It is difficult to solve nonlinear differential equations and, in general, it is often more difficult to get an analytic approximate than a numerical one of a given nonlinear oscillatory system [2]. There is a large variety of approximate methods for the determination of solutions of nonlinear second-order dynamical systems including perturbation [3], standard and modified Lindstedt–Poincaré [4–6], variational [7], variational iteration [8], homotopy perturbation [9–12], harmonic balance [13–17] methods, etc. Surveys of the literature with numerous references and useful bibliography and a review of these methods can be found in detail in [2] and [18]. In this Letter we apply the first-order harmonic balance method to obtain analytic approximate solutions for the relativistic oscillator. This is a procedure for determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation. This method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required.

When the energy of a simple harmonic oscillator is such that the velocities become relativistic, the simple harmonic motion (linear oscillations) at low energy becomes anharmonic (nonlinear oscillations) at high energy [19]. Due to this fact we have

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considered the parentheses around the “an” in the title of this Letter. Then, the strength of the nonlinearity increases as the total relativistic energy increases, and at the non-relativistic limit the oscillator becomes linear. Mickens [20] showed that all the solutions to the relativistic (an)harmonic oscillator are periodic and determined a method for calculating analytical approximations to its solutions. Mickens considered the first-order harmonic balance method, but he did not apply the technique correctly and the first analytical approximate frequency he obtained is not the correct one.

2. Nonlinear differential equation for the relativistic oscillator

The governing non-dimensional nonlinear differential equation of motion for the relativistic oscillator is [20]

$$\frac{d^2x}{dt^2} + \left[1 - \left(\frac{dx}{dt}\right)^2\right]^{3/2} x = 0 \tag{1}$$

where x and t are dimensionless variables. The even power term in Eq. (1), $(dx/dt)^2$, acts like the powers of coordinates in that it does not cause a damping of the amplitude of oscillations with time. Therefore, Eq. (1) is an example of a generalized conservative system [1]. At the limit when $(dx/dt)^2 \ll 1$, Eq. (1) becomes $(d^2x/dt^2) + x \approx 0$ the oscillator is linear and the proper time τ ($d\tau = \sqrt{1-(dx/dt)^2}\,dt$) becomes equivalent to the coordinate time t to this order.

Introducing the phase space variable (x, y), Eq. (1) can be written as follows

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\left(1-y^2\right)^{3/2} x \tag{2}$$

and the trajectories in phase space are given by solutions to the first order, ordinary differential equation

$$\frac{dy}{dx} = -\frac{(1-y^2)^{3/2}x}{y}. \tag{3}$$

As Mickens pointed out, since the physical solutions of both Eq. (1) and Eq. (3) are real, the phase space has a “strip” structure [20], i.e.,

$$-\infty < x < +\infty \quad \text{and} \quad -1 < y < +1. \tag{4}$$

Then unlike the usual non-relativistic harmonic oscillator, the relativistic oscillator is bounded in the y variable. This is due to the fact that the non-dimensional variable y is related with the relativistic relation $\beta = v/c$, where v is the velocity of the particle and c the velocity of light. In the relativistic case, the condition $-c < v(t) < +c$ must be met, and so we obtain $-1 < y(t) < +1$. Mickens proved that all the trajectories to Eq. (3) are closed in the open region of phase space given by Eq. (4) and then all the physical solutions to Eq. (1) are periodic. However, unlike the usual (non-relativistic) harmonic oscillator, the relativistic (an)harmonic oscillator contains higher-order multiples of the fundamental frequency.

In order to apply the harmonic balance method, we make a change of variable, $y \to u$, such that $-\infty < u < +\infty$, as follows

$$y = \frac{u}{\sqrt{1+u^2}} \tag{5}$$

and the corresponding second-order nonlinear differential equation for u is

$$\frac{d^2u}{dt^2} + \frac{u}{\sqrt{1+u^2}} = 0. \tag{6}$$

We consider the following initial conditions in Eq. (7)

$$u(0) = B \quad \text{and} \quad \frac{du}{dt}(0) = 0. \tag{7}$$

Eq. (6) is an example of conservative nonlinear oscillatory system having irrational form for the restoring force. This is a conservative nonlinear oscillatory system and all the motions corresponding to Eq. (6) are periodic [20], the system will oscillate symmetric bounds $[-B, B]$, and the angular frequency and corresponding periodic solution of the nonlinear oscillator are dependent on the amplitude B.

The main objective of this Letter is to solve Eq. (6) by applying the first-order harmonic balance method and to compare the approximate frequency obtained with the exact one and with another approximate frequency obtained by applying the method of harmonic balance to the same oscillatory system but rewriting Eq. (6) in a way suggested previously by Mickens [20]. Comparing with the approximate solution obtained by this last procedure, the approximate frequency derived here is more accurate with respect to exact solution. The errors of the resulting frequency are reduced and the maximum relative error is less than 1.6% for the complete range of oscillation amplitudes, including the limiting cases of amplitude approaching zero and infinity.