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Method of self-similar factor approximants

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Abstract

The method of self-similar factor approximants is completed by defining the approximants of odd orders, constructed from the power series with the largest term of an odd power. It is shown that the method provides good approximations for transcendental functions. In some cases, just a few terms in a power series make it possible to reconstruct a transcendental function *exactly*. Numerical convergence of the factor approximants is checked for several examples. A special attention is paid to the possibility of extrapolating the behavior of functions, with arguments tending to infinity, from the related asymptotic series at small arguments. Applications of the method are thoroughly illustrated by the examples of several functions, nonlinear differential equations, and anharmonic models.

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1. Introduction

The problem of reconstructing functions from their asymptotic series is widely known to be of great importance for a variety of applications [1]. Probably, the most often used technique allowing for such a reconstruction is that based on Padé approximants [2], though there also exist other more complicated methods [3].

Recently, a novel approach has been advanced for reconstructing functions from the related asymptotic series, called the method of *self-similar factor approximants* [4–6]. The form of these approximants is derived by means of the self-similar approximation theory [7–15]. The factor approximants were shown [4–6] to be more general and more accurate than Padé approximants. However, several points in the method of factor approximants [4–6] have not been investigated.

The most important point, which has remained unclear, is how to construct the factor approximants of *odd orders*? The problem is that the standard procedure [4–6] requires that the generic series, the factor approximants are built from, be of even order. So that the large amount of information, contained in the odd orders of the series, could not be properly used. Here we advance a uniform approach for treating both odd as well as even orders of power series and we demonstrate, by a number of examples, good performance of this general method.

Another, rather nontrivial, question is why the factor approximants could provide high accuracy for *transcendental functions*, obtained from their asymptotic series. We give an explanation for this effect, which is unique for the resummation methods based on asymptotic series. Recall that Padé approximants, having the structure of rational functions, are usually not so good for approximating transcendental functions. Moreover, we show that some transcendental functions can be reconstructed, by means of the factor approximants, *exactly* from a few terms of an asymptotic series.

In the previous papers [4–6] on the method of self-similar factor approximants, we mainly considered the low-order approximants of even order, because of which their numerical convergence could not be properly analyzed. Now, we possess both even and odd orders of these approximants, and we study

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their numerical convergence for various cases, calculating high-order approximants, up to 20th order.

We pay a special attention to the possibility of extrapolating physically motivated functions to the whole range of their variables. We demonstrate that the method of self-similar factor approximants can serve as a tool for defining the values of functions at infinity from their asymptotic expressions near zero.

Summarizing, the main results of the present Letter are as follows:

(i) The method of self-similar factor approximants is extended by constructing the factor approximants of odd orders, in addition to those of even orders.

(ii) It is shown that transcendental functions can be well approximated. A unique feature of the method is that some transcendental functions can be reconstructed exactly.

(iii) The method allows for the extrapolation of solutions to nonlinear differential equations from asymptotically small to finite variables.

(iv) The characteristics of quantum anharmonic models can be extrapolated from the region of asymptotically small coupling parameter to the whole range of the latter, including the case of the coupling parameter tending to infinity.

(v) For all considered examples, the factor approximants of high orders are calculated, demonstrating the existence of numerical convergence.

2. Self-similar factor approximants

Let us be interested in defining a real function $f(x)$ of a real variable $x \in \mathbb{R}$. The extension of the method to complex-valued functions of complex variables is also possible, but, first, let us consider a slightly simpler case of real functions and variables. Suppose, we know only the behavior of the function at asymptotically small values of $x \to 0$, where we can get the sequence $\{f_k(x)\}$ of the expansions

$$f_k(x) = \sum_{n=0}^{k} a_n x^n, \tag{1}$$

with $k = 0, 1, 2, \ldots$. The sequence, generally, can be divergent. It is known [1] that a function $f(x)$, analytic in the vicinity of $x = 0$, uniquely defines its asymptotic series. The converse is not always true. But in what follows, we consider the situation, when there is a one-to-one correspondence between the function and its asymptotic series, since solely then the problem of reconstructing functions from their expansions acquires sense.

Without the loss of generality, we may assume that in expansion (1),

$$a_0 = f_k(0) = f_0(x) = 1. \tag{2}$$

Really, if instead of expansion (1), we would have

$$f^{(k)}(x) = f^{(0)}(x) \sum_{n=0}^{k} a_n x^n,$$

with a given function $f^{(0)}(x)$, then we could immediately return to Eq. (1) defining

$$f_k(x) \equiv \frac{f^{(k)}(x)}{f^{(0)}(x) a_0}.$$

The self-similar factor approximants of even orders $k = 2p = 2, 4, 6, \ldots$ are given [4–6] by the form

$$f_{2p}^*(x) = \prod_{i=1}^{p} (1 + A_i x)^{n_i}, \tag{3}$$

whose parameters A_i and n_i are obtained from the re-expansion procedure, that is, by expanding Eq. (3) in powers of x and comparing the results with the given expansion (1), equating the like-order terms. This accuracy-through-order procedure yields the set of equations

$$\sum_{i=1}^{p} n_i A_i^n = B_n \quad (n = 1, 2, \ldots, 2p), \tag{4}$$

with the right-hand sides

$$B_n \equiv \frac{(-1)^{n-1}}{(n-1)!} \lim_{x \to 0} \frac{d^n}{dx^n} \ln f_k(x). \tag{5}$$

In each approximation, the parameters A_i, n_i, and B_n, of course, depend on the approximation number $k = 2p$. However, in order to avoid too cumbersome notation, we do not mark explicitly this dependence, which is assumed to be evident.

Each factor in form (3) contains two parameters, A_i and n_i. This is why these approximants could be straightforwardly defined only for the even-order expansions $f_{2p}(x)$. But how should we proceed having odd-order expansions (1) with $k = 2p + 1 = 1, 3, 5, \ldots$? For the latter, we could write the form

$$f_{2p+1}^*(x) = \prod_{i=1}^{p+1} (1 + A_i x)^{n_i}, \tag{6}$$

with the parameters satisfying the set of equations

$$\sum_{i=1}^{p+1} n_i A_i^n = B_n \quad (n = 1, 2, \ldots, 2p + 1). \tag{7}$$

But the problem is that form (6) contains $2p + 2$ unknown A_i and n_i, while only $2p + 1$ equations of set (7) are available. One equation is lacking.

To overcome this problem, let us notice that expression (6) is invariant under the scaling transformation

$$x \to \lambda x, \qquad A_i \to \lambda^{-1} A_i. \tag{8}$$

Then, taking $\lambda \to A_1^{-1}$ and using the renotation $A_i/A_1 \to A_i$, we come to the same form (6), but with

$$A_1 = 1 \quad (k = 2p + 1). \tag{9}$$

Complementing the set of $2p + 1$ equations (7) by the scaling condition (9), we get $2p + 2$ equations for $2p + 2$ unknowns. In this way, we now can construct the factor approximants of odd orders.