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# A new method based on the harmonic balance method for nonlinear oscillators

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**Abstract**

The harmonic balance (HB) method as an analytical approach is widely used for nonlinear oscillators, in which the initial conditions are generally simplified by setting velocity or displacement to be zero. Based on HB, we establish a new theory to address nonlinear conservative systems with arbitrary initial conditions, and deduce a set of over-determined algebraic equations. Since these deduced algebraic equations are not solved directly, a minimization problem is constructed instead and an iterative algorithm is employed to seek the minimization point. Taking Duffing and Duffing-harmonic equations as numerical examples, we find that these attained solutions are not only with high degree of accuracy, but also uniformly valid in the whole solution domain.


*Keywords:* Nonlinear oscillator; Harmonic balance method; Duffing equation; Duffing-harmonic equation; Arbitrary initial conditions; Uniformly valid

## 1. Introduction

The harmonic balance (HB) method is one of the most widely used methods for nonlinear oscillators. The basic physics is to transform the problem under consideration into a set of nonlinear algebraic equations by truncated Fourier series [1–3]. Importantly, it is not only valid for both weakly and strongly nonlinear oscillators, but also provides uniformly valid solutions in the whole solution domain for the crucial equations such as the Duffing equation. Generally, it is relatively easy to carry out the lower HB method, and the attained solutions are accurate enough for some nonlinear problems, especially for weakly nonlinear ones. However, it is necessary to solve a set of complicated nonlinear algebraic equations for the higher HB method. This is a big drawback of the HB method. Therefore, researchers tried to overcome the limitation by combining HB with other methods. For instance, Cheung and Lau [4–6] proposed the incremental harmonic balance (IHB) method. Further, the IHB method has been developed and successfully applied in various nonlinear problems [7–9]. Lim and Wu [10,11] proposed an analytical approach by combining linearization and HB. They deduced the linear algebraic equations instead of nonlinear ones using the linearization prior to proceeding the harmonic balancing [12]. Interestingly, this method was successfully used to solve the Duffing equation [11], the Duffing-harmonic equation [10], the nonlinear oscillation of the conservative system having inertia and static nonlinearities [13] and the damped van der Pol equation [14]. Additionally, Lim et al. [15] introduced the Newton's method to simplify the complexity of HB method. However, most of the studies above all adopt simple initial conditions [10–15], which seems to introduce new limitations to these developed methods above. For this issue, in the present paper, we try to solve the nonlinear conservative systems

$$\ddot{x} + f(x) = 0 \tag{1}$$

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subject to any given initial conditions

$$x(0) = A, \quad \dot{x}(0) = B, \tag{2}$$

where the superscript denotes the differentiation with respect to $t$. Without loss of generality, in our case, the circumstance that $f(x)$ is an odd function is discussed.

It is well known that Nayfeh [1] applied the HB method to solve the equations with polynomial nonlinearities by supposing

$$x = \sum_{m=1}^{M} A_m \cos(m\omega t + m\beta_0), \tag{3}$$

where $\omega$ is the angular frequency of periodic solution. The procedure is setting the coefficients of the harmonics lower than $M+1$ to be zeroes, thus resulting in a set of algebraic equations with respect to $A_m$ and $\omega$.

Substituting Eq. (3) into Eq. (2) results in

$$\sum_{m=1}^{M} A_m \cos(m\beta_0) = A, \qquad -\sum_{m=1}^{M} m\omega A_m \sin(m\beta_0) = B. \tag{4}$$

If $A_m$ and $\omega$ are determined, Eqs. (4) are over-determined equations about $\beta_0$. Generally, there is no suitable $\beta_0$ to ensure Eq. (3) satisfying Eqs. (2). Then, one feasible way is to eliminate one equation, i.e., $A_m$, $\omega$ and $\beta_0$ can be all obtained by setting the coefficients of the harmonics lower than $M$ instead of $M+1$ to zeros. However, it is very difficult to do this because these equations contain very complex nonlinearities. On the other hand, considering the simple initial condition (i.e., $B=0$), the approximate solution is described as [10–15]

$$x = x_0 + \Delta x, \tag{5}$$

where $x_0$ is the initial approximation and $\Delta x$ is correction part, with

$$x_0 = A\cos\tau, \qquad \Delta x = \sum_{k=1}^{K} c_k\bigl\{\cos\bigl[(2k-1)\tau\bigr] - \cos\bigl[(2k+1)\tau\bigr]\bigr\}, \quad \tau = \omega t. \tag{6}$$

Based on Eqs. (6), we can see that Eq. (5) can always satisfy the initial conditions that $x(0) = A$, $\dot{x}(0) = 0$.

Note that, in our study, we firstly employ the HB method that is different from Eq. (3) in form to solve Eqs. (1) and (2). Then, we propose a new method based on the idea of Eq. (5). Our studies show that the present method reveals one easier way for implementing the HB method because it can be carried out without solving any nonlinear algebraic equations.

## 2. The harmonic balance method

The solution of Eqs. (1) and (2) is periodic with the frequency of $\omega$. Introducing a new variable that $\tau = \omega t$, then we can transform Eqs. (1) and (2) into

$$\omega^2 x'' + f(x) = 0, \quad x(0) = A, \quad x'(0) = \frac{B}{\omega}, \tag{7}$$

where the superscript denotes the differentiation with respect to $\tau$. As $f(x)$ is odd, the $(N+1)$th HB solution can be described as

$$x_{N+1} = \sum_{k=1}^{N+1} \bigl\{c_k \cos\bigl[(2k-1)\tau\bigr] + s_k \sin\bigl[(2k-1)\tau\bigr]\bigr\}. \tag{8}$$

Substituting Eq. (8) into Eq. (7), expanding Eq. (7) as Fourier series about $\tau$, and letting the coefficients of $\{\cos\tau, \sin\tau, \dots, \cos[(2N+1)\tau], \sin[(2N+1)\tau]\}$ be zeros, then we have

$$\begin{aligned}
&\Gamma_{ck}(c,s;\omega) = 0, \qquad \Gamma_{sk}(c,s;\omega) = 0, \quad k = 1,2,\dots,N+1,\\
&\sum_{k=1}^{N+1} c_{2k-1} = A, \qquad \sum_{k=1}^{N+1}(2k-1)s_{2k-1} = \frac{B}{\omega},
\end{aligned} \tag{9}$$

where the vector $c$ and $s$ are defined as $c = [c_1, c_2, \dots, c_{N+1}]^{\mathrm{T}}$ and $s = [s_1, s_2, \dots, s_{N+1}]^{\mathrm{T}}$, respectively. There are $2N+3$ unknowns and $2(N+2)$ equations in Eqs. (9). Thus, we have to eliminate one equation. By removing the equation of $\Gamma_{s,N+1} = 0$, Eqs. (9) become

$$\Gamma_{ck}(c,s;\omega) = 0, \quad k = 1,2,\dots,N+1; \qquad \Gamma_{sk}(c,s;\omega) = 0, \quad k = 1,2,\dots,N;$$