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Exact solutions of Laplace equation by homotopy-perturbation and Adomian decomposition methods

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Abstract

In this work, two powerful analytical methods, called homotopy-perturbation method (HPM) and Adomian decomposition method (ADM) are introduced to obtain the exact solutions of Laplace equation with Dirichlet and Neumann boundary conditions. The results obtained by these methods are then compared with variational iteration method (VIM). The comparison among these methods shows that although the numerical results of these methods are the same, HPM is much easier, more convenient and efficient than ADM and VIM.

Keywords: Laplace equation; Homotopy-perturbation method; Adomian decomposition method; Variational iteration method

1. Introduction

Partial differential equations which arise in real-world physical problems are often too complicated to be solved exactly. And even if an exact solution is obtainable, the required calculations may be too complicated to be practical, or it might be difficult to interpret the outcome. Very recently, some promising approximate analytical solutions are proposed, such as Exp-function method [1,2], Adomian decomposition method [3–7], variational iteration method [8–10] and homotopy-perturbation method [11–16]. Other methods are reviewed in Refs. [17,18].

HPM is the most effective and convenient one for both linear and nonlinear equations. This method does not depend on a small parameter. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$, which is considered as a "small parameter". HPM has been shown to effectively, easily and accurately solve a large class of linear and nonlinear problems with components converging rapidly to accurate solutions. HPM was first proposed by He [11] and was successfully applied to various engineering problems [19–21].

Recently, VIM is applied for exact solutions of Laplace equation [22]. The aim of this work is to employ HPM and ADM to obtain the exact solutions for Laplace equations and to compare the results with those of VIM. Different from ADM, where specific algorithms are usually used to determine the Adomian polynomials, HPM handles linear and nonlinear problems in a simple manner by deforming a difficult problem into a simple one.

The Laplace equation is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics. The two-dimensional Laplace equation has the following form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

or

$$\nabla^2 u = 0, \tag{2}$$

where ∇^2 is the Laplacian.

The Dirichlet boundary conditions for Laplace's equation consist in finding a solution of u on domain D such that on the boundary of D is equal to some given function [23,24]. One physical interpretation of this problem which arises in heat equations is as follows: fix the temperature on the boundary of

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the domain and wait until the temperature in the interior does not change anymore; the temperature distribution in the interior will then be given by the solution to the corresponding Dirichlet problem.

The Neumann boundary conditions for Laplace's equation specify not the function itself on the boundary of D, but its normal derivative [23,24]. Physically, this is similar to the construction of a potential for a vector field whose effect is known at the boundary of D alone.

In this work, four Laplace equations, two with Dirichlet boundary conditions and two with Neumann boundary conditions, are studied.

2. Fundamentals of the homotopy-perturbation method

To illustrate the basic ideas of this method, we consider the following equation [11]:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3}$$

with the boundary condition of

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{4}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Eq. (3) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega. \tag{5}$$

Homotopy perturbation structure is shown as follows:

$$H(v, p) = (1-p)\big[L(v) - L(u_0)\big] + p\big[A(v) - f(r)\big] = 0, \tag{6}$$

where

$$v(r, p): \Omega \times [0, 1] \to R. \tag{7}$$

In Eq. (6), $p \in [0, 1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (6) can be written as a power series in p, as following:

$$v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots \tag{8}$$

and the best approximation for solution is

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots. \tag{9}$$

The above convergence is discussed in [11].

3. Implementation of HPM to the Laplace equation

In order to assess the advantages and accuracy of HPM, we consider the following examples.

Example 1. Consider the two-dimensional Laplace equation

$$u_{xx} + u_{yy} = 0, \quad x > 0,\ y < \pi, \tag{10}$$

subject to the boundary conditions of

$$u(0, y) = 0, \qquad u(\pi, y) = \sinh\pi \cos y,$$
$$u(x, 0) = \sinh x, \qquad u(x, \pi) = -\sinh x. \tag{11}$$

In order to solve Eq. (10), using HPM, we construct the following homotopy for this equations:

$$H(v, p) = (1-p)v_{xx} + p(v_{xx} + v_{yy}) = 0. \tag{12}$$

Substituting v from Eq. (8) into Eq. (12) and rearranging based on powers of p-terms, we can obtain:

$$p^0: \quad \frac{\partial^2 v_0}{\partial x^2} = 0, \tag{13}$$

$$p^1: \quad \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} = 0, \tag{14}$$

$$p^2: \quad \frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} = 0, \tag{15}$$

$$p^3: \quad \frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} = 0. \tag{16}$$

Solving Eq. (13), we obtain:

$$v_0(x, y) = x \cos y. \tag{17}$$

The zeroth approximation $v_0(x, y)$ satisfies three boundary conditions when considering $\sinh x \approx x$. Solving Eqs. (14)–(16) we obtain:

$$v_1(x, t) = \frac{1}{3!}x^3 \cos y, \tag{18}$$

$$v_2(x, t) = \frac{1}{5!}x^5 \cos y, \tag{19}$$

$$v_3(x, t) = \frac{1}{7!}x^7 \cos y. \tag{20}$$

The solution of Eq. (10) when $p \to 1$ will be as follows:

$$u(x, y) = \cos y\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots\right). \tag{21}$$

Therefore, the exact solution of $u(x, y)$ in closed form is

$$u(x, y) = \sinh x \cos y \tag{22}$$

which is the same as that obtained by VIM [22].

Example 2. Consider the two-dimensional Laplace equation

$$u_{xx} + u_{yy} = 0, \quad x > 0,\ y < \pi, \tag{23}$$

subject to the boundary conditions of

$$u(0, y) = \sin y, \qquad u(\pi, y) = \cosh\pi \sin y,$$
$$u(x, 0) = 0, \qquad u(x, \pi) = 0. \tag{24}$$

Proceeding as before, we construct the following homotopy:

$$H(v, p) = (1-p)v_{xx} + p(v_{xx} + v_{yy}) = 0. \tag{25}$$