Available online at www.sciencedirect.com
ScienceDirect
Physics Letters A 367 (2007) 88–94
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Application of homotopy analysis method to fractional KdV–Burgers–Kuramoto equation ☆

Lina Song *, Hongqing Zhang

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, PR China

Received 20 January 2007; accepted 26 February 2007

Available online 1 March 2007

Communicated by A.R. Bishop

Abstract

In the Letter, homotopy analysis method that developed for integer-order differential equation is directly extended to derive explicit and numerical solutions of nonlinear fractional differential equation for the first time. The fractional derivatives are described in the Caputo sense. To our knowledge, the Letter represents the first available numerical solutions of the fractional KdV–Burgers–Kuramoto equation.

PACS: 03.40.Kf; 03.65.Ge

Keywords: Homotopy analysis method; Factional KdV–Burgers–Kuramoto equation; Fractional calculus

1. Introduction

In the past decades, both mathematicians and physicists have devoted considerable effort to the study of explicit solutions to nonlinear integer-order differential equation. Many powerful methods have been presented [1–29]. Among them, the homotopy analysis method (HAM) [17–22] provides an effective procedure for explicit and numerical solutions of a wide and general class of differential systems representing real physical problems. Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in the considered equation. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a possibility to analyze strongly nonlinear problems. This method has been successfully applied to solve many types of nonlinear problems by others [23–29]. However, the application of HAM only circumscribes integer-order differential equation. Here, we introduce the method to nonlinear fractional differential equation.

In recent years, considerable interest in fractional differential equation has been stimulated due to their numerous applications in the areas of physics and engineering [30]. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are well described by fractional differential equation [31–33]. The solution of fractional differential equation is much involved. In general, there exists no method that yields an exact solution for fractional differential equation. Only approximate solutions can be derived using linearization or perturbation method.

The aim of this Letter is to directly extend the HAM to consider the numerical solution of the fractional KdV–Burgers–Kuramoto (KBK) equation with space-fractional derivatives of the form

$$\frac{\partial u}{\partial t}+u\frac{\partial^{\alpha}u}{\partial x^{\alpha}}+a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^3 u}{\partial x^3}+c\frac{\partial^4 u}{\partial x^4}=0,\quad t>0,\ 0<\alpha\leqslant 1,\tag{1.1}$$

☆ The work is partially supported by the National Key Basic Research Project of China under the Grant No. 2004CB318000.

* Corresponding author.
E-mail address: songlina1981@yahoo.com.cn (L. Song).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2007.02.083

where a, b, c are constants and α is parameter describing the order of the fractional space-derivatives. The function $u(x,t)$ is assumed to be a causal function of space, i.e. vanishing for $t<0$ and $x<0$. The fractional derivatives are considered in the Caputo sense [34]. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha=1$, Eq. (1.1) reduces to the classical nonlinear KBK equation. More important, above procedure is just an algebraic algorithm and can be easily applied in the symbolic computation system Maple.

Although there are a lot of studies for the classical KBK equation and some profound results have been established, it seems that detailed studies of the nonlinear fractional differential equation are only beginning. According to our knowledge, this Letter represents the first available numerical solution of the fractional KBK equation with space-fractional derivatives.

The Letter has been organized as follows. In Section 2, a brief review of the theory of fractional calculus will be given to fix notation and provide a convenient reference. In Section 3, we give analysis of the HAM. In Section 4, we extend the application of the HAM to construct numerical solutions for the fractional KBK equation. Conclusions are presented in Section 5.

2. Preliminaries and notations

In this section, let us recall essentials of fractional calculus first. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. There are many books [30–33] that develop fractional calculus and various definitions of fractional integration and differentiation, such as Grünwald–Letnikov's definition, Riemann–Liouville's definition, Caputo's definition and generalized function approach. For the purpose of this Letter the Caputo's definition of fractional differentiation will be used, taking the advantage of Gaputo's approach that the initial conditions for fractional differential equation with Caputo's derivatives take on the traditional form as for integer-order differential equation.

Definition 2.1. Caputo's definition of the fractional-order derivative is defined as

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}\frac{f^{(n)}\tau}{(t-\tau)^{\alpha+1-n}}d\tau \quad \bigl(n-1<\mathrm{Re}(\alpha)\leqslant n,\ n\in N\bigr), \tag{2.1}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex, a is the initial value of function f. In this Letter only real and positive α will be considered. For the Caputo's derivative we have

$$D^{\alpha}C=0 \quad (C \text{ is a constant}), \tag{2.2}$$

$$D^{\alpha}t^{\beta}=\begin{cases}0 & (\beta\leqslant\alpha-1),\\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha} & (\beta>\alpha-1).\end{cases} \tag{2.3}$$

Similar to integer-order differentiation, Caputo's fractional differentiation is a linear operation:

$$D^{\alpha}\bigl(\lambda f(t)+\mu g(t)\bigr)=\lambda D^{\alpha}f(t)+\mu D^{\alpha}g(t), \tag{2.4}$$

where λ, μ are constants, and satisfies the so-called Leibnitz rule:

$$D^{\alpha}\bigl(g(t)f(t)\bigr)=\sum_{k=0}^{\infty}\binom{\alpha}{k}g^{(k)}(t)D^{\alpha-k}f(t), \tag{2.5}$$

if $f(\tau)$ is continuous in $[a,t]$ and $g(\tau)$ has $n+1$ continuous derivatives in $[a,t]$.

In the Letter, we consider Eq. (1.1), where the unknown function $u=u(x,t)$ is assumed to be a causal function of space, and the fractional derivative is taken in Caputo sense as follows:

Definition 2.2. For n to be the smallest integer that exceeds α, the Caputo space-fractional derivative operator of order $\alpha>0$ is defined as

$$D_x^{\alpha}u(x,t)=\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-\tau)^{n-\alpha-1}\frac{\partial^n u(\tau,t)}{\partial\tau^n}d\tau & \text{if } n-1<\alpha<n,\\ \frac{\partial^n u(x,t)}{\partial x^n} & \text{if } \alpha=n\in N.\end{cases} \tag{2.6}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult Ref. [33].