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Physics Letters A 367 (2007) 156–161
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Wave scattering by small particles in a medium

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Received 14 February 2007; received in revised form 20 February 2007; accepted 21 February 2007

Available online 23 February 2007

Communicated by V.M. Agranovich

Abstract

Wave scattering is considered in a medium in which many small particles are embedded. Equations for the effective field in the medium are derived when the number of particles tends to infinity.

PACS: 03.65.Nk; 43.20.+g; 62.30.+d

Keywords: Wave scattering; Small particles; Many-body scattering problem

1. Introduction

Assume that waves in the medium are described by the equation

$$Lu := \big(a_{ip}(x)u_{,p}\big)_i + k^2 n(x)u = 0 \quad \text{in } \mathbb{R}^3, \tag{1}$$

where over the repeated indices summation is understood, $u_{,p} := \frac{\partial u}{\partial x_p}$, and the Green function, satisfying the radiation condition, solves the equation:

$$LG = -\delta(x-y) \quad \text{in } \mathbb{R}^3. \tag{2}$$

If there are M particles D_m, placed in the medium, situated in a bounded domain D, outside of which

$$a_{ip}(x) = \delta_{ip}, \quad n(x) = 1, \quad x \in D' := \mathbb{R}^3 \setminus D, \tag{3}$$

where $a_{ip}(x)$, $n(x)$ are C^2-smooth functions, δ_{ip} is the Kronecker symbol, and the ellipticity condition holds:

$$c_1 \sum_{p=1}^{3} |t_p|^2 \leqslant \sum_{i,p=1}^{3} a_{ip} t_p \bar{t_i} \leqslant c_2 \sum_{p=1}^{3} |t_p|^2, \quad c_1 > 0,$$

where $t \in \mathbb{C}^3$ is an arbitrary vector, then the scattering problem consists of solving the equation

$$L\mathcal{U} = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m, \tag{4}$$

$$\mathcal{U}|_S = 0 \quad \text{on } S_m,\ 1 \leqslant m \leqslant M, \tag{5}$$

$$\mathcal{U} = \mathcal{U}_0 + \sum_{m=1}^{M} \int_{S_m} G(x,s)\sigma_m(s)\,ds. \tag{6}$$

Here $L\mathcal{U}_0 = 0$, u_0 is the scattering solution in the absence of particles, i.e., if $M = 0$.

By Ramm's lemma [3, p. 257], one can define the scattering solution $\mathcal{U}_0$ in the absence of small particles by the relation:

$$G(x,y) = g(y)\mathcal{U}_0(x,\beta)\big[1 + o(1)\big], \quad |y| \to \infty,\ \frac{y}{|y|} = -\beta, \tag{7}$$

where $g(y) := \frac{e^{ik|y|}}{4\pi|y|}$. We assume that $ka \ll 1$, where $a = \frac{1}{2}\max_{1 \leqslant m \leqslant M} \operatorname{diam} D_m$.

The aim of this Letter is to develop a general approach to wave scattering in a medium in which many small particles are embedded. Smallness of the particles is understood in the sense $ka \ll 1$. The functions $n(x)$ and $a_{ip}(x)$ are assumed practically

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2007.02.076

constant on the scale of the wavelength

$$k\big(|\nabla n| + |\nabla a_{ip}|\big) \ll 1.$$

We generalize the approach developed in [4–7]. Earlier works are [1,10,11], to mention a few. Let us briefly discuss the difference between our work and the well-known work [1]. In [1] the scatterers are assumed to be points, the "scattering coefficients" (i.e., the scattering amplitudes, corresponding to a single scatterer) are assumed known, while in our work analytical formulas for these "coefficients" are derived, so that these "coefficients" are known and depend on the shapes of the particles; the boundary conditions on the surfaces of the particles are taken into account in our theory, but do not appear in [1] because there is no boundary of a point scatterer. The small particles in our theory are not necessarily randomly distributed. They, for example, can be oriented similarly. On the other hand, our theory is applicable to the case when the particles are randomly distributed.

Our basic result is a formula for the wave field in the medium in which small particles are embedded. This field solves Eq. (4)–(6) and satisfies the radiation condition at infinity:

$$\frac{\partial(\mathcal{U}-\mathcal{U}_0)}{\partial r} - ik(\mathcal{U}-\mathcal{U}_0) = o\left(\frac{1}{r}\right), \qquad r := |x| \to \infty. \tag{8}$$

We assume that

$$d \gg a, \qquad ka \ll 1, \tag{9}$$

where $d = \min_{m\neq j} \operatorname{dist}(D_m, D_j)$, and the dist denotes the distance between two sets. Near any point $x \in D$, such that $\min_{1\leqslant m\leqslant M} \operatorname{dist}(x, D_m) \gg a$, one calculates the wavefield $\mathcal{U}$ by the formula

$$\mathcal{U}(x) = \mathcal{U}_0(x) + \sum_{m=1}^{M} G(x, x_m) \times \int_{S_m} \big[1 + ik\nu\cdot(s - x_m)\big]\sigma_m(s)\,ds, \tag{10}$$

where $\frac{\nu}{|\nu|} \approx \frac{x-x_m}{|x-x_m|}$, and ν depends on x_m, x, and on the functions $n(x)$, $a_{ip}(x)$. The term $ik\nu\cdot(s - x_m)$ comes from the formulas

$$\int_{S_m} G(x,s)\sigma_m(s)\,ds = G(x,x_m)\left\{\int_{S_m}\sigma_m(s)\,ds + \int_{S_m}\frac{[G(x,s)-G(x,x_m)]}{G(x,x_m)}\sigma_m(s)\,ds\right\},$$

$$\frac{G(x,s)-G(x,x_m)}{G(x,x_m)} = \frac{\int_0^1 \nabla_y G(x, x_m + \tau(s-x_m))\cdot(s-x_m)\,d\tau}{G(x,x_m)} := ik\nu\cdot(s-x_m),$$

where $\nu = \nu(x, x_m)$, $\frac{\nu}{|\nu|} = \frac{x_m - x}{|x_m - x|}$. One may consider ν as a known vector because $G(x, y)$ is known.

In a generic case, when $Q_m := \int_{S_m}\sigma_m(s)\,ds \neq 0$, the assumption $ka \ll 1$ allows one to neglect the term $ik\nu\cdot(s-x_m) = O(ka)$ in (10) and to write (10) as

$$\mathcal{U}(x) = \mathcal{U}_0(x) + \sum_{m=1}^{M} G(x,x_m)Q_m, \qquad Q_m = \int_{S_m}\sigma_m\,ds. \tag{11}$$

If $Q_m = 0$, then the term $ik\nu\cdot s$ cannot be neglected. We discuss this case in Section 3. A physical example of such a case is the scattering by acoustically hard particles when the boundary condition is the Neumann one: $\frac{\partial\mathcal{U}}{\partial N}|_{S_m} = 0$, $1 \leqslant m \leqslant M$.

2. General methodology

Let us first assume that (11) is applicable and calculate Q_m. In a neighborhood of S_j one has the exact boundary condition (5), which can be written as:

$$\int_{S_j} G(s,t)\sigma_j(t)\,dt = -\Big(\mathcal{U}_0(s) + \sum_{m\neq j} G(s,x_m)Q_m\Big) := u_e(s), \tag{12}$$

where $s \in S_j$ and u_e is the effective field acting on D_j. The error one makes using Eq. (12) is of the order $O(ka + \frac{a}{d})$, so in the limit $ka \to 0$ and $\frac{a}{d} \to 0$ Eq. (23) becomes exact. The derivation of Eq. (23) is based on Eqs. (12), (16)–(18), and (22).

Our basic assumption, related to assumptions (9), is:

We assume that $u_e(s)$ is practically constant on the distances of order a.

As $|s-t| \to 0$, one has

$$G(s,t) = \frac{1}{4\pi|s-t|}\big[1 + O(ka)\big], \qquad |s-t| \to 0, \tag{13}$$

where $s, t \in S_j$ and we have assumed for simplicity that $a_{ij} = \delta_{ij}$. In the general case one replaces the function $g_0(s,t) := \frac{1}{4\pi|s-t|}$ on the surface S_j by the fundamental solution of the operator $\sum_{i,p=1}^{3} a_{ip}(x_j)\frac{\partial^2}{\partial x_i\partial x_p}$, which can be written explicitly and analytically:

$$G(x,y) = \frac{1}{4\pi\sqrt{\det(a_{ip})}}\frac{1}{[a_{ip}^{(-1)}(x_i - y_i)(x_p - y_p)]^{1/2}},$$

where the matrix $a_{ip}^{(-1)}$ is inverse of the matrix $a_{ip}(x)$, x_i is the ith Cartesian component of the vector x (not to be confused with the vector $x_j = x$).

If $a_{ip} = \delta_{ip}$, then G solves the equation

$$G(x,y) = g_0(x,y) + k^2\int_D g(x,t)n(t)G(t,y)\,dt, \qquad g_0(x,y) = \frac{1}{4\pi|x-y|}. \tag{14}$$