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Localization estimation and global vs. local information measures

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Abstract

The maximum entropy principle is one of the great ideas of the last 50 years, with a multitude of applications in many areas of science. Its main ingredient is an information measure. We show that global and local information measures provide *different* types of physical information, which requires handling them with some care. The concomitant differences are illustrated with reference to the problem of localization in phase space, placing emphasis on some details of the smoothing of Wigner functions, as described in [G. Manfredi, M.R. Feix, Phys. Rev. E 62 (2000) 4665]. Our discussion is made in terms of a special version of Fisher's information measure, called the shift-invariant one.

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1. Introduction

The Maximum Entropy Principle (MaxEnt) constitutes one of the great ideas of the second half of the last century, with manifold applications not only to several subdisciplines of physics but of other sciences as well. The main MaxEnt ingredient is an "entropy", or more properly, an information measure that, upon extremizing, yields the probability distribution that describes the properties of the system at hand. In the wake of MaxEnt's success, a new paradigm for physics research has been advanced by Wheeler about 15 years ago [1,2] that tries to ascertain to what an extent physical theories can be derived from information theory. Great progress has been made. Electromagnetism, classical physics, quantum mechanics, general relativity, and many other formalisms have been re-derived from an information basis (see, for instance, [3]). The "quantum information–computation revolution" that is taking place right now [4] is another factor that motivates researchers to try to delve deeper into different aspects of the handling of information-theoretic tools, and in particular, into the properties of Fisher's information measure (FIM) [5]. In order to introduce the FIM-concept we begin by reminding the reader of an important fact, namely, that information measures come in two flavors: global and local.

Global information measures (SP) are those obeying Kinchin's axioms for information theory [6]. They depend exclusively on a probability distribution (PD). The Shannon–Boltzmann–von Neumann entropy is the foremost example, although we do not use it here. *Local* information measures, on the other hand, depend also on various derivatives of the PD. Fisher's measure is the best known example. Current Fisher-literature is abundant. Much has been learned with reference to its quantum mechanical applications, specially with regards to bounds to various relevant quantities and relations with the uncertainty principle (see, for instance, [7] and references therein). However, bridges between quantal phase-space treatments and Fisher's measure have not yet received comparable attention, and it is one of our present goals to contribute to remedying this situation. It is not always clear, from a physical viewpoint, whether

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local or global measures are to be employed in a given scenario. In many areas, mostly related to statistical physics, they compete. When the relevant physics is governed by a differential equation (e.g., Dirac's equation for relativistic quantum mechanics), local measures are preferable [3].

In such a vein we wish to show, for local vs. global measures, how different their respective information content may be and will discuss the issue with reference to an important physical problem: Phase-space localization. We consider a Husimi–Fock approach and a Wigner one [8–11].

We will in this Letter revisit the matter from the point of view of Fisher's information (a local measure) I and thereby translate the smoothing process in terms of *localization estimation in phase-space*. In this respect, as stated above, a vital difference between I and S_P will emerge, which constitutes our present leit motif.

2. Fisher's information and linear entropy S_P

Fisher's I is an important information measure, advanced by Fisher in the twenties (a detailed study can be found in Refs. [3,12]), with the idea of assessing *intrinsic accuracy* in statistical estimation theory [3,12]. When associated with *translations* of a one-dimensional observable x, with corresponding probability density $P(x)$, the so-called shift-invariant I is given by (not the most general I-definition) [3,12]

$$I = I[P] \equiv I_P = \int \mathrm{d}x\, P(x)\big\{\mathrm{d}\ln P(x)/\mathrm{d}x\big\}^2. \tag{1}$$

Note that the derivative operation $\mathrm{d}/\mathrm{d}x$ causes I to be a "local" measure [3,12].

The most important I-application in estimation theory is the Cramer–Rao lower bound [3,12]

$$I_P e_P^2 \geqslant 1; \quad e_P^2 \equiv x\text{'s-variance}. \tag{2}$$

Fisher's measure is additive [3]: If x and p are independent variables, $I(x,p) = I(x) + I(p)$, a useful concept if we are interested in estimating location (x,p) in phase-space [13]. We may note that in quantum mechanics probability amplitude $\psi(x)$ depends on probability amplitude $\tilde{\psi}(p)$ (as Fourier transform mates), so in this sense the x and p spaces are not independent. However, individual data values x, p are indeed independent samples from the PDs $|\psi(x)|^2$ and $|\tilde{\psi}(p)|^2$. This independence property will be used below.

The global measure S_P to be used here for comparing with Fisher's I is not the usual Shannon–Boltzmann entropy. S_P is, instead, related to the quantum concept of "purity" T of a mixed state. For a probability distribution $P(x)$ this is defined by [14]

$$T = \int \mathrm{d}x\, P(x)^2, \qquad S_P = 1 - T \quad (0 \leqslant S_P \leqslant 1). \tag{3}$$

$R = T^{-1}$ is called the participation ration. If P arises from a mixed quantum state, R tells us about the number of pure states entering the mixture. S_P vanishes in the case of complete information, while it equals unity for total ignorance. As stated, S_P will be the global entropic measure used in the rest of the Letter. We will call it "linear entropy" following current usage [11,14]. In dealing with density matrices S_P exhibits the decisive practical advantage of being computable without recourse to diagonalization, which becomes mandatory in the Boltzmann–Shannon instance because of its logarithmic nature.

3. Husimi semiclassical distribution

Smoothing the Wigner function W one can overcome the problem of its being non-positive definite [11]. In particular, smoothing a Wigner function with the minimum uncertainty Gaussian $G(x,p)$ gives rise to Husimi semiclassical probability distribution functions (PDF) [10]. In Ref. [11] arguments have been given, for didactic purposes, to the effect that, if the function to be smoothed is itself a Gaussian W_o, then the variance of the smoothing function should be close to that of W_o in order to *optimize the information-content* of the ensuing $\bar{W}$ distribution. For a given density operator $\hat{\rho}$, the Husimi PDFs are of the form $Q(\alpha) = \langle\alpha|\hat{\rho}|\alpha\rangle$, with $|\alpha\rangle$ a coherent state whose eigenvalue equation (for the annihilation operator $\hat{a}$) reads $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. We introduce now some notation concerning minimum uncertainty Gaussians, such as those representing the ground state of a harmonic oscillator of Hamiltonian $\hat{H}_o = \hbar\omega[\hat{a}^\dagger\hat{a} + 1/2]$. We have

$$\begin{aligned}
&\hat{a} = i(2\hbar\omega m)^{-1/2}\hat{p} + (m\omega/2\hbar)^{1/2}\hat{x},\\
&\alpha = x/2\sigma_x + ip/2\sigma_p,\\
&\sigma_x = \sqrt{\hbar/2m\omega}, \qquad \sigma_p = \sqrt{\hbar m\omega/2}, \qquad 2\sigma_x\sigma_p = \hbar.
\end{aligned} \tag{4}$$

Variances σ are evaluated for a minimum uncertainty Gaussian. Coherent states span Hilbert's space, constitute an over-complete basis and obey the completeness rule [15,16] $\int(\mathrm{d}^2\alpha/\pi) \times |\alpha\rangle\langle\alpha| = \iint(\mathrm{d}x\,\mathrm{d}p/2\pi\hbar)|x,p\rangle\langle x,p| = 1$. The Husimi function $Q(x,p) = Q(\alpha) = \langle\alpha|\hat{\rho}|\alpha\rangle$ is normalized in the fashion $\iint(\mathrm{d}x\,\mathrm{d}p/2\pi\hbar)Q(x,p) = 1$ [10].

3.1. Cramer–Rao relations

It is clear from (4) that $|\alpha|^2 = x^2/4\sigma_x^2 + p^2/4\sigma_p^2$, and for any symmetrical (in both x and p) PDF $P(x,p)$ one has $\langle|\alpha|^2\rangle_P = \langle x^2/4\sigma_x^2\rangle_P + \langle p^2/4\sigma_p^2\rangle_P$. It is well known that (virial theorem)

$$\big\langle|\alpha|^2\big\rangle_P = 2\big\langle x^2/4\sigma_x^2\big\rangle_P = 2\big\langle p^2/4\sigma_p^2\big\rangle_P, \tag{5}$$

and, since $\langle x\rangle_P = \langle p\rangle_P = 0 \Rightarrow \langle\alpha\rangle_P = 0$, we have

$$\sqrt{\big\langle x^2\big\rangle_P - \langle x\rangle_P^2} = \Delta_{(P)}x = \sqrt{2}\sigma_x\sqrt{\big\langle|\alpha|^2\big\rangle_P}, \tag{6}$$

$$\sqrt{\big\langle p^2\big\rangle_P - \langle p\rangle_P^2} = \Delta_{(P)}p = \sqrt{2}\sigma_p\sqrt{\big\langle|\alpha|^2\big\rangle_P}. \tag{7}$$

These show that the (square-root of) the localization variance of any arbitrary PDF $P(x,p)$ is related to that of the minimum uncertainty Gaussian $(\sigma_x\sigma_p)$ according to

$$(\Delta x\Delta p)_P = 2\sigma_x\sigma_p\big\langle|\alpha|^2\big\rangle_P = \hbar\big\langle|\alpha|^2\big\rangle_P \tag{8}$$

since $\sigma_x\sigma_p = \hbar/2$. Summing up, for any arbitrary symmetrical PDF $P(x,p)$, $(\Delta x\Delta p)_P$ differs from $\langle|\alpha|^2\rangle_P$ just in a factor $\hbar$.