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Exponential convergence of a spectral projection of the KdV equation

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Abstract

It is shown that a spectral approximation of the Korteweg–de Vries equation converges exponentially fast to the true solution if the Fourier basis is used and if the solution is analytic in a fixed strip about the real axis. Computations are carried out which show that the exponential convergence rate can be achieved in practice.

Keywords: Spectral methods; Convergence rate; Solitary waves

1. Introduction

In this Letter, consideration is given to the convergence of a spectral projection of the periodic Korteweg–de Vries (KdV) equation

$$\partial_t u + u\partial_x u + \frac{1}{a^3}\partial_x^3 u = 0, \tag{1.1}$$

where a is a positive real number. Evidently, for the KdV equation on the line $\mathbb{R}$, a can be taken to be equal to 1, as it can be scaled out using the scaling $u(x,t) = \frac{1}{a}v(ax,t)$. However, as we are considering the equation on the interval $[0, 2\pi]$ with periodic boundary conditions, it will be convenient to leave a unspecified.

The KdV equation has been useful as a model equation in a variety of contexts, including the study of water waves, particle physics and flow in blood vessels [1–5], just to name a few. The discovery by Zabusky and Kruskal of the elastic interaction of solitary waves [6], and the subsequent formulation of a solution algorithm by way of solving an inverse-scattering problem [3,7], excited interest in the equation from both the mathematical and physical point of view. Along with the nonlinear Schrödinger equation, the KdV equation has become a paradigm for nonlinear wave equations featuring competing nonlinear and dispersive effects. Since the discovery by Cooley and Tukey of a fast algorithm to compute the discrete Fourier transform [8], spectral methods based on the fast Fourier transform have become a popular choice for the spatial discretization of nonlinear partial differential equations. In particular, in wave propagation problems, spectral projection has been widely used in connection with the Fourier basis.

The convergence of spectral projections of the KdV equation was proved by Maday and Quarteroni [9]. In particular, it was shown that if the solution $u(x,t)$ of (1.1) is smooth, then spectral convergence is achieved. That is, if u_N denotes the approximate solution with N grid points, then there is a constant λ_T, such that

$$\sup_{t\in[0,T]} \big\| u(\cdot,t) - u_N(\cdot,t)\big\| \leqslant \lambda_T N^{-m},$$

for any positive integer m. This estimate shows that the convergence rate is higher than any algebraic rate. It was announced in [10] that if the solution u is analytic in a strip about the real axis, then the convergence rate is in fact exponential. Thus there exist constants Λ_T and σ_T, depending on T, such that

$$\sup_{t\in[0,T]} \big\| u(\cdot,t) - u_N(\cdot,t)\big\| \leqslant \Lambda_T N e^{-\sigma_T N}. \tag{1.2}$$

In this Letter, a proof of this estimate will be given. Moreover, a numerical study will be conducted to show that this result is

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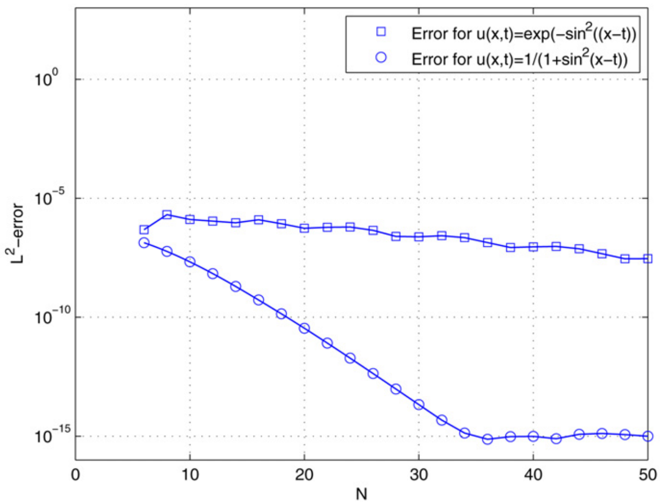

Fig. 1. It appears that the exponential convergence rate which holds for analytic solutions is more advantageous than the traditional spectral convergence which holds for smooth, but not analytic solutions.

indeed achievable in practice. To indicate the significance of the improvement, Fig. 1 shows the result of computing approximate solutions of the inhomogeneous equation

$$\partial_t u + u\partial_x u + \frac{1}{a^3}\partial_x^3 u = f.$$

First, the function $u(x,t) = e^{-\sin^2(x-t)}$ is used as the exact solution. Note that this function is smooth, but not analytic. The values shown as boxes in Fig. 1 are the L^2-error between the exact solution u and the computed approximation u_N. For this function, spectral convergence is achieved. Next, we use the function $u(x,t) = \frac{1}{1+\sin^2(x-t)}$ which is analytic. The resulting L^2-errors are shown as circles in Fig. 1, and it appears that the exponential convergence rate is by far superior to the spectral convergence rate. It should be noted that exponential convergence for spectral approximation schemes for evolution equations has been studied earlier. The exponential convergence of Galerkin schemes for parabolic equations has been previously advocated by Ferrari and Titi [11] and proved for the Ginsburg–Landau equation by Doelman, Jones, Margolin and Titi [12,13]. There have not been any previous results in this direction for dispersive equations like the KdV equation. However, Matthies and Scheel have used the analytic Gevrey norms to be defined in the next section in connection with dispersive equations in another context [14].

The plan of this Letter is as follows. In Section 2, we establish the relevant mathematical notation. In Section 3, the spectral approximation is defined, and the convergence estimate (1.2) is proved. Finally, in Section 4, numerical computations are shown to elucidate the result of Section 3. As it will turn out, the numerical observations match the theoretical predictions superbly.

2. Notation

The results in this Letter hold for solutions of (1.1) which are real-analytic functions of the spatial variable x. To quantify the domain of analyticity, we use the class of periodic analytic Gevrey spaces as introduced by Foias and Temam in [15]. For $\sigma > 0$, we define the Gevrey norm $\|\cdot\|_{G_\sigma}$ by

$$\|f\|_{G_\sigma}^2 = \sum_{k\in\mathbb{Z}} e^{2\sigma\sqrt{1+|k|^2}} \left|\hat{f}(k)\right|^2,$$

where the Fourier coefficients $\hat{f}(k)$ of the function f, periodic on the interval $[0, 2\pi]$ are defined by

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)\,dx.$$

A Paley–Wiener type argument shows that functions in the space G_σ are analytic in a strip of width 2σ about the real axis. Similarly, the usual periodic Sobolev spaces are given by the norm

$$\|f\|_{H^s}^2 = \sum_{k\in\mathbb{Z}} \left(1+|k|^2\right)^s \left|\hat{f}(k)\right|^2.$$

In particular, for $s = 0$, the space $L^2(0, 2\pi)$ appears. For simplicity, the L^2-norm is written without any subscript, so that $\|f\| = \|f\|_{H^0}$. In the sequel, we will have occasion to use the inner product on this space, given by

$$(f, g) = \int_0^{2\pi} f(x)\overline{g(x)}\,dx.$$

Note also that for functions $f \in H^s$ with $s > \frac{1}{2}$, we have the Sobolev inequality, namely

$$\sup_x \left|f(x)\right| \leqslant C\|f\|_{H^s}$$

for some constant C.

3. Spectral projection

The spectral projection is achieved by solving a discrete set of ordinary differential equations in a finite-dimensional space. For this purpose, the subspace

$$S_N = \left\{e^{ikx} \mid k \in \mathbb{Z}, -N \leqslant k \leqslant N\right\}$$

of $L^2(0, 2\pi)$ is commonly used in connection with the Fourier basis. The self-adjoint operator P_N denotes the orthogonal projection from L^2 onto S_N, defined by

$$P_N f(x) = \sum_{-N\leqslant k\leqslant N} e^{ikx}\hat{f}(k).$$

Observe that P_N may also be characterized by the property that, for any $f \in L^2$, $P_N f$ is the unique element in S_N such that

$$(P_N f, \phi) = (f, \phi), \tag{3.1}$$

for all $\phi \in S_N$. Using a straightforward calculation, the following inequality can be proved [18]

$$\|f - P_N f\|_{H^r} \leqslant N^r e^{-\sigma N}\|f\|_{G_\sigma}, \tag{3.2}$$