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General N-box problem

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Abstract

The paradoxical properties of the N-box problem ($N \geqslant 3$) are dispelled by the straightforward exercise of constructing, for an *arbitrary* pre-selected state with N nonzero orthogonal components, the corresponding (unique) post-selected state such that this pair leads to the same properties. A new and interesting N-box problem is presented.

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1. Introduction

Fifteen years ago Aharonov and Vaidman [1] analyzed "a curious situation in which *one* particle can be found with certainty in $N-1$ (!) separate boxes".[1] They considered a quantum particle prepared at $t = t_{\rm in}$ in the initial single-particle state

$$|\Psi_{\rm in}\rangle = N^{-1/2}\big[|1\rangle + \cdots + |N-1\rangle + |N\rangle\big] \tag{1}$$

and successfully post-selected at time $t = t_{\rm f}$ in the final state

$$|\Psi_{\rm f}\rangle = \big(N^2 - 3N + 3\big)^{-1/2}\big[|1\rangle + \cdots + |N-1\rangle - (N-2)|N\rangle\big], \tag{2}$$

where $\{|i\rangle \equiv |\psi_i\rangle \mid i = 1, 2, \ldots, N\}$ with $N \geqslant 3$ is an orthonormal basis set. When the spatial supports of the wave functions $\{\psi_i(x) \equiv \langle x|\psi_i\rangle\}$ are mutually disjoint then finding the particle in the state $|j\rangle$ is identified with finding it somewhere in the jth box defined by the spatial support of $\psi_j(x)$. Aharonov and Vaidman used the Aharonov–Bergman–Lebowitz (ABL) rule [2] to calculate the probability of finding the particle in the jth box at an intermediate time $t = T$ between $t_{\rm in}$ and $t_{\rm f}$ when only the jth box was probed for the presence of the particle by a strong measurement of the observable associated with the projector $\hat{P}_j \equiv |j\rangle\langle j|$ having eigenvalues $P_j = 0$ (particle not found) and 1 (particle found). Assuming that the time evolution was negligible for $[t_{\rm in}, T - 0^+]$ and $[T + 0^+, t_{\rm f}]$, they showed that whenever the postselection was successful the particle had been found in the jth box with certainty for any value of j except $j = N$. Hence, according to Aharonov and Vaidman [3], "This single particle is, in some sense, simultaneously in $N-1$ boxes".

This "curious situation" has generated such interest and controversy that the simplest case ($N = 3$) is frequently referred to as "the 3-box paradox". In this case

$$|\Psi_{\rm in}\rangle = 3^{-1/2}\big(|1\rangle + |2\rangle + |3\rangle\big), \qquad |\Psi_{\rm f}\rangle = 3^{-1/2}\big(|1\rangle + |2\rangle - |3\rangle\big). \tag{3}$$

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[1] In order to maintain a consistent notation throughout, N has been changed here to $N-1$.

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According to the ABL rule, for the successfully postselected subensemble the probability that the particle was found in box 1 at time $t = T$ when only that box was "opened" in a strong "position" measurement is given by [3]

$$\mathrm{Prob}(P_1 = 1|\Psi_{\mathrm{in}}, \Psi_{\mathrm{f}}; \hat{P}_1) = \frac{|\langle\Psi_{\mathrm{f}}|\hat{P}_1|\Psi_{\mathrm{in}}\rangle|^2}{|\langle\Psi_{\mathrm{f}}|\hat{P}_1|\Psi_{\mathrm{in}}\rangle|^2 + |\langle\Psi_{\mathrm{f}}|[\hat{1} - \hat{P}_1]|\Psi_{\mathrm{in}}\rangle|^2} = \frac{|\langle\Psi_{\mathrm{f}}|1\rangle\langle 1|\Psi_{\mathrm{in}}\rangle|^2}{|\langle\Psi_{\mathrm{f}}|1\rangle\langle 1|\Psi_{\mathrm{in}}\rangle|^2 + |\langle\Psi_{\mathrm{f}}|2\rangle\langle 2|\Psi_{\mathrm{in}}\rangle + \langle\Psi_{\mathrm{f}}|3\rangle\langle 3|\Psi_{\mathrm{in}}\rangle|^2}$$
$$= \frac{|1/3|^2}{|1/3|^2 + |1/3 - 1/3|^2} = 1. \tag{4}$$

Similarly, $\mathrm{Prob}(P_2 = 1|\Psi_{\mathrm{in}}, \Psi_{\mathrm{f}}; \hat{P}_2) = 1$. Aharonov and Vaidman [3] regarded this as having one particle in two places simultaneously "even in a stronger sense than it is in a double slit experiment".[2] They also emphasized the contextuality evident in the fact that the corresponding probability that the particle was found in boxes 1 and 2 probed as a unit is less than 1, i.e. $\mathrm{Prob}(P_{1\cup 2} = 1|\Psi_{\mathrm{in}}, \Psi_{\mathrm{f}}; \hat{P}_{1\cup 2}) < 1$ with $\hat{P}_{1\cup 2} \equiv |1\rangle\langle 1| + |2\rangle\langle 2|$.

In an attempt to better understand the "sense" in which one particle can be in several disjoint spatial regions simultaneously, we considered the N-box problem for an arbitrary initial state $|\Psi_{\mathrm{in}}\rangle$. Section 2.1 presents the analysis for the case of a strong intermediate measurement. Sections 2.2 and 2.3 consider the case of a weak intermediate measurement. In Section 3, results for both strong and weak intermediate measurements are presented for another interesting N-box problem. Brief concluding remarks are made in Section 4.

In [3], Aharonov and Vaidman stressed the fact that, for a situation in which both a preselection was made at t_{in} and a postselection was made at t_{f}, the present moment of time, t_{now}, must be greater than t_{f}, i.e. $t_{\mathrm{in}} < T < t_{\mathrm{f}} < t_{\mathrm{now}}$. This means that it is important to use the correct tense in discussing the N-box problem. In particular, using the present tense for the intermediate measurement can be misleading.

2. The N-box problem

In this section, starting with any normalized initial state with $N \geqslant 3$ nonzero orthogonal components, we construct the corresponding post-selected state such that the main predicted statistical results for each of the intermediate strong measurements considered in the N-box problem are exactly the same, for successful postselection, as those obtained by Aharonov and Vaidman for the pair of states (1) and (2). For completeness, the case of weak intermediate measurements is also considered.

2.1. Strong measurement case

Consider the normalized initial single-particle state

$$|\Psi_{\mathrm{in}}\rangle \equiv \big|\Psi(t = t_{\mathrm{in}})\big\rangle = d_{\mathrm{in}}^{-1}\sum_{i=1}^{N} c_i|i\rangle \quad (N \geqslant 3); \qquad d_{\mathrm{in}} \equiv \left[\sum_{i=1}^{N} |c_i|^2\right]^{1/2}. \tag{5}$$

It is assumed without loss of generality that $|c_i| \neq 0$ or $\neq \infty$ $(i = 1, 2, \ldots, N)$ but is otherwise arbitrary. To emphasize the fact that N different experimental arrangements are involved, we consider N ensembles—one ensemble for each of the N different experiments contemplated—with each member of each ensemble prepared in the same state $|\Psi_{\mathrm{in}}\rangle$ at the initial time $t = t_{\mathrm{in}}$. Suppose that for the jth ensemble a strong measurement was made at an intermediate time $t = T$ of the observable associated with the projection operator $\hat{P}_j \equiv |j\rangle\langle j|$. There are, of course, two possible results: eigenvalue $P_j = 1$ was obtained with probability

$$\mathrm{Prob}(P_j = 1|\Psi_{\mathrm{in}}; \hat{P}_j) = \big|\langle j|\Psi_{\mathrm{in}}\rangle\big|^2 = d_{\mathrm{in}}^{-2}|c_j|^2 \tag{6}$$

and the state collapsed to

$$|\Psi_j\rangle \equiv |j\rangle; \tag{7}$$

eigenvalue $P_j = 0$ was obtained with probability

$$\mathrm{Prob}(P_j = 0|\Psi_{\mathrm{in}}; \hat{P}_j) = d_{\mathrm{in}}^{-2} d_{\mathrm{not}\,j}^{2}; \qquad d_{\mathrm{not}\,j} \equiv \left[\sum_{i=1; i\neq j}^{N} |c_i|^2\right]^{1/2} \tag{8}$$

and the state instead collapsed to

$$|\Psi_{\mathrm{not}\,j}\rangle \equiv d_{\mathrm{not}\,j}^{-1} \sum_{i=1; i\neq j}^{N} c_i|i\rangle. \tag{9}$$

[2] Kirkpatrick presented a three slit implementation of the 3-box system in [4] and concluded that 'its paradoxical behavior is as ordinary (to the extent that any atomic Young system may be considered "ordinary"!) as a double-slit interference apparatus'.