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Physics Letters A 359 (2006) 428–437
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Simple criteria for the SLOCC classification ☆

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Received 20 April 2006; received in revised form 30 June 2006; accepted 3 July 2006

Available online 13 July 2006

Communicated by P.R. Holland

Abstract

We put forward an alternative approach to the SLOCC classification of three-qubit and four-qubit systems. By directly solving matrix equations, we obtain the relations satisfied by the amplitudes of states. The relations are readily tested since in them only addition, subtraction and multiplication occur.

PACS: 03.65.Bz; 03.65.Hk

Keywords: Entanglement; Quantum computing; Quantum information; Separability; SLOCC

1. Introduction

Entanglement plays a key role in quantum computing and quantum information. If two states can be obtained from each other by means of local operations and classical communication (LOCC) with nonzero probability, we say that two states have the same kind of entanglement [1]. It is well known that a pure entangled state of two-qubits can be locally transformed into an *EPR* state. For multipartite systems, there are several inequivalent forms of entanglement under asymptotic LOCC [2]. Recently, many authors have investigated the equivalence classes of three-qubit states specified by stochastic local operations and classical communication (SLOCC) [3–13]. Dür et al. showed that for pure states of three-qubits there are six different classes of the entanglement under SLOCC, out of which there are two inequivalent types of genuine tripartite entanglement [4]. They put forward a principled method to distinguish the six classes from each other by calculating the ranks of the reduced density matrices and the minimal product decomposition [4]. For example, they pointed out that if a state of a three-qubit system with $r(\rho_A) = r(\rho_B) = r(\rho_C) = 2$ has 2 (respectively 3) product terms in its minimal product decomposition under SLOCC, then the state is equivalent to state $|GHZ\rangle$ (respectively $|W\rangle$). However, no criterion has so far been proposed for the minimal number of product decomposition terms under SLOCC [3,4,15]. In a more recent paper, Verstraete et al. [9] considered the entanglement classes of four-qubits under SLOCC and concluded that there exist nine families of states corresponding to nine different ways of entanglement. In these previous papers, the authors just put forward some principled rules of classifying the entangled states. It requires complicated calculations when these principled rules are applied to real states. It will be quite useful if a feasible approach can be found. Coffman et al. presented "concurrence" and defined 3-tangle [14]. Later, Dür et al. utilized 3-tangle and the local entropies S_A, S_B and S_C to

☆ The paper was supported by NSFC (Grant No. 60433050), the basic research fund of Tsinghua university No: JC2003043 and partially by the state key lab. of intelligence technology and system.
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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2006.07.004

describe SLOCC classification of three qubits [4]. Miyake discussed the onionlike classification of SLOCC orbits and proposed the SLOCC equivalence classes using the orbits [10]. Rajagopal and Rendell gave the conditions for the fullseparability and the biseparability [13].

Here, we present an alternative approach to classifying the entanglement of three-qubits, and then generalize the case of four-qubits. We will give simple criteria of distinguishing the entanglement classes from each other simply by checking the relations satisfied by the amplitudes of the states.

2. The SLOCC classification of a three-qubit system

We first discuss the system comprising three qubits A, B, C. The states of a three-qubit system can be generally expressed as

$$|\psi\rangle = a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + a_4|100\rangle + a_5|101\rangle + a_6|110\rangle + a_7|111\rangle.$$

Two states $|\psi\rangle$ and $|\psi'\rangle$, are equivalent under SLOCC if and only if there exist invertible local operators α, β and γ such that

$$|\psi\rangle = \alpha \otimes \beta \otimes \gamma |\psi'\rangle, \tag{1}$$

where the local operators α, β and γ can be expressed as 2×2 invertible matrices

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix}.$$

We consider the following six classes, respectively.

2.1. The $|GHZ\rangle$ class

Let $|\psi'\rangle \equiv |GHZ\rangle$, i.e.

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\big(|000\rangle + |111\rangle\big). \tag{2}$$

Substituting Eq. (2) into Eq. (1), we get

$$\begin{aligned} a_0 &= (\alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2)/\sqrt{2}, & a_1 &= (\alpha_1\beta_1\gamma_3 + \alpha_2\beta_2\gamma_4)/\sqrt{2}, \\ a_2 &= (\alpha_1\beta_3\gamma_1 + \alpha_2\beta_4\gamma_2)/\sqrt{2}, & a_3 &= (\alpha_1\beta_3\gamma_3 + \alpha_2\beta_4\gamma_4)/\sqrt{2}, \\ a_4 &= (\alpha_3\beta_1\gamma_1 + \alpha_4\beta_2\gamma_2)/\sqrt{2}, & a_5 &= (\alpha_3\beta_1\gamma_3 + \alpha_4\beta_2\gamma_4)/\sqrt{2}, \\ a_6 &= (\alpha_3\beta_3\gamma_1 + \alpha_4\beta_4\gamma_2)/\sqrt{2}, & a_7 &= (\alpha_3\beta_3\gamma_3 + \alpha_4\beta_4\gamma_4)/\sqrt{2}. \end{aligned}$$

By calculating $a_i a_j - a_k a_l$, where $i + j = k + l$ and $i \oplus j = k \oplus l$, where $\oplus$ is a addition modulo two for binary numbers, we obtain the following equations:

$$\begin{aligned} a_2a_4 - a_0a_6 &= \gamma_2\gamma_1(\alpha_1\alpha_4 - \alpha_3\alpha_2)(\beta_2\beta_3 - \beta_4\beta_1)/2, \\ a_3a_5 - a_1a_7 &= \gamma_4\gamma_3(\alpha_1\alpha_4 - \alpha_3\alpha_2)(\beta_2\beta_3 - \beta_4\beta_1)/2, \\ a_0a_7 - a_3a_4 &= -(\alpha_1\alpha_4 - \alpha_2\alpha_3)(-\gamma_4\beta_4\gamma_1\beta_1 + \beta_3\gamma_3\beta_2\gamma_2)/2, \\ a_1a_6 - a_2a_5 &= -(-\alpha_3\alpha_2 + \alpha_1\alpha_4)(-\gamma_2\beta_4\gamma_3\beta_1 + \beta_3\gamma_1\beta_2\gamma_4)/2. \end{aligned}$$

From the above equations, we further obtain

$$(a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4)^2 - 4(a_2a_4 - a_0a_6)(a_3a_5 - a_1a_7) = \frac{1}{4}(\alpha_1\alpha_4 - \alpha_2\alpha_3)^2(-\gamma_4\gamma_1 + \gamma_3\gamma_2)^2(-\beta_4\beta_1 + \beta_2\beta_3)^2. \tag{3}$$

$|\psi\rangle$ is equivalent to $|GHZ\rangle$ under SLOCC, if and only if the invertible operators α, β and γ exist. From Eq. (3), we may immediately conclude that the necessary and sufficient condition of $|\psi\rangle$ being equivalent to $|GHZ\rangle$ is

$$(a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4)^2 - 4(a_2a_4 - a_0a_6)(a_3a_5 - a_1a_7) \neq 0. \tag{4}$$

It is not hard to verify that

$$\begin{aligned} &\big(a_0a_7 - a_2a_5 + (a_1a_6 - a_3a_4)\big)^2 - 4(a_2a_4 - a_0a_6)(a_3a_5 - a_1a_7) \\ &\quad = \big(a_0a_7 - a_3a_4 - (a_1a_6 - a_2a_5)\big)^2 - 4(a_1a_4 - a_0a_5)(a_3a_6 - a_2a_7) \\ &\quad = \big(a_0a_7 - a_2a_5 - (a_1a_6 - a_3a_4)\big)^2 - 4(a_0a_3 - a_1a_2)(a_4a_7 - a_5a_6). \end{aligned} \tag{5}$$