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Transient growth for streak-streamwise vortex interactions

Lina Kim, Jeff Moehlis *

Department of Mechanical Engineering, University of California, Santa Barbara, CA 93106, USA

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Abstract

We analyze transient growth due to the linear interaction between streaks and streamwise vortices. We obtain initial perturbations which give optimal initial and total energy growth, characterize the dependence of the dynamics on the initial distribution of perturbation energy, and compare with results from pseudospectra analysis.

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1. Introduction

Shear flows are fluid flows which are non-homogeneous with a mean shear. Turbulent shear flows are of great fundamental physical and mathematical interest because [3]: (i) Turbulence is found both experimentally and in numerical simulations for values of the Reynolds number well below the value at which the laminar state loses stability [1], and (ii) the governing partial differential equations possess numerous branches of (unstable) steady or traveling states consisting of wavy streamwise vortices and streaks that arise in saddle-node bifurcations [4–7]. Such solutions have recently been detected experimentally [3, 8], but their relevance to turbulence remains unclear.

It has been suggested that transient energy growth provides a good basis for understanding property (i) (e.g. [2,23]). Such transient growth can significantly amplify small perturbations to the laminar state which can trigger non-linear effects that lead to sustained turbulence via the self-sustaining process identified in [12,13]. In this Letter, we analyze transient growth due to the linear interaction of the streak and streamwise vortex modes from the nine-mode model from [10,14]. This is a model for sinusoidal shear flow, which obeys the non-dimensional equations

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \frac{1}{Re}\nabla^2\mathbf{u} + \mathbf{F}(y),$$
$$\nabla\cdot\mathbf{u} = 0, \tag{1}$$

where the Reynolds number and body force are defined as

$$Re = \frac{U_0 d}{2\nu}, \qquad \mathbf{F}(y) = \frac{\sqrt{2}\pi^2}{4Re}\sin(\pi y/2)\hat{\mathbf{e}}_x, \tag{2}$$

U_0 is the characteristic velocity and ν is the kinematic viscosity. The free-slip boundary conditions

$$u_y = 0, \qquad \frac{\partial u_x}{\partial y} = \frac{\partial u_z}{\partial y} = 0 \tag{3}$$

are imposed at $y = \pm 1$, and the flow is assumed periodic in the streamwise (x) and spanwise (z) directions, with lengths L_x and L_z, respectively. The laminar profile for sinusoidal shear flow,

$$\mathbf{U}(y) = \sqrt{2}\sin(\pi y/2)\hat{\mathbf{e}}_x \tag{4}$$

is linearly stable for all Re [1]. In the following, we let $\alpha = 2\pi/L_x$, $\beta = \pi/2$, and $\gamma = 2\pi/L_z$. Although difficult to obtain experimentally, sinusoidal shear flow represents a non-trivial shear flow which is amenable to analytical treatment; it is hoped that the knowledge gained from the present analysis will be

* Corresponding author. Tel.: +1 805 893 7513; fax: +1 805 893 8651.
E-mail addresses: lina@engineering.ucsb.edu (L. Kim), moehlis@engineering.ucsb.edu (J. Moehlis).

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helpful for characterizing other shear flows such as plane Couette flow, boundary layer flow, Poiseuille flow, and pipe flow.

In Section 2, a geometric interpretation of transient growth due to the interaction between streaks and streamwise vortices is given. Then, the details of how such transient growth depends on initial conditions, Re, and aspect ratio are derived. Furthermore, the neutral transient growth curve, below which *no* initial condition gives transient energy growth, is found and discussed. In Section 3, the transient energy growth is interpreted using pseudospectra analysis, where a lower bound for the maximum attainable energy is obtained using Kreiss' theorem. We will see that the analysis in Section 2 gives a sharper characterization of the transient growth than that in Section 3. Our conclusions are given in Section 4.

2. Transient growth for the streak-streamwise vortex interaction

The matrix M arising from the linearization of the nine-mode model from [10] about the laminar state is non-normal, i.e., $MM^T \neq M^T M$. This suggests that even though its eigenvalues are all strictly negative for all Re, corresponding to linear stability of the laminar state, it might be possible to have transient growth of energy which could trigger non-linear effects that sustain turbulence [2,25]. In this section, a detailed analysis is conducted for the transient growth which occurs for the 2×2 block of M that corresponds to the linear evolution of the amplitudes a_2 and a_3, which give the amplitudes of the streamwise invariant streak and streamwise vortex modes

$$\mathbf{u}_2 = \begin{pmatrix} \frac{4}{\sqrt{3}} \cos^2(\pi y/2)\cos(\gamma z) \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{u}_3 = \frac{2}{\sqrt{4\gamma^2 + \pi^2}} \begin{pmatrix} 0 \\ 2\gamma \cos(\pi y/2)\cos(\gamma z) \\ \pi \sin(\pi y/2)\sin(\gamma z) \end{pmatrix}, \tag{5}$$

respectively. We focus on this interaction because it gives the strongest transient energy growth compared to the other interactions of the linearized nine-dimensional model. Furthermore, streaks and streamwise vortices are dominant structures in numerical simulations and are related to unstable steady solutions of the Navier–Stokes equations [4–6,9]. Finally, they are related to optimal perturbations [17] and are the most energetically excited structures of the linearized Navier–Stokes equations with forced input and can be explained as input–output resonances of frequency responses [19].

A Galerkin projection onto these modes gives the linear system

$$\begin{pmatrix} \dot{a}_2 \\ \dot{a}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} b & c \\ 0 & d \end{pmatrix}}_{M_{23}} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}, \tag{6}$$

$$b = -\frac{\frac{4\beta^2}{3} + \gamma^2}{Re} = \mathcal{O}(Re^{-1}), \qquad c = -\frac{\sqrt{3/2}\beta\gamma}{\sqrt{\beta^2+\gamma^2}} = \mathcal{O}(Re^0),$$

$$d = -\frac{\beta^2+\gamma^2}{Re} = \mathcal{O}(Re^{-1}). \tag{7}$$

Here and elsewhere we give the scaling behavior for large Re. The laminar state corresponds to $a_2 = a_3 = 0$; the stability of the laminar state with respect to streak and streamwise vortex perturbations follows from the fact that the eigenvalues b and d of M_{23} are negative. The exact solution to (6) is readily shown to be

$$a_2(t) = a_{20}e^{bt} + \frac{c}{d-b}a_{30}\left(e^{dt} - e^{bt}\right), \quad a_3(t) = a_{30}e^{dt}. \tag{8}$$

For this system, the energy is defined to be $E(t) = (a_2(t))^2 + (a_3(t))^2$. We note that (6) also arises in the linearization of the eight-mode model from [13] and the uncoupled-mode model from [15] about the laminar state, with the same Re dependence of b, c, d but with different values.

2.1. Geometric interpretation of transient energy growth

The solution (8) can be rewritten in a form which allows an instructive geometric interpretation of transient energy growth, namely

$$\boldsymbol{a}(t) = \left(a_2(t), a_3(t)\right) = \underbrace{\mathbf{v}_1 b_{10} e^{bt}}_{\mathbf{s}_1(t)} + \underbrace{\mathbf{v}_2 b_{20} e^{dt}}_{\mathbf{s}_2(t)}, \tag{9}$$

where formulas for b_{10} and b_{20} in terms of a_{20} and a_{30} are readily obtained, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are the normalized eigenvectors for M_{23}. Since M_{23} is non-normal, $\mathbf{v}_1$ and $\mathbf{v}_2$ are non-orthogonal; for example, for the values $L_z = 1.2\pi$ and $Re = 400$ studied below, they are almost anti-parallel. For the related system of plane Couette flow, these parameters correspond to the minimal flow unit, the smallest domain which is found numerically to sustain turbulence [16]. A small-amplitude initial condition is thus the superposition of two very large-amplitude components; i.e., $|\mathbf{s}_1(0)|$ and $|\mathbf{s}_2(0)|$ are large, as sketched in the left panels of Fig. 1. For the linear system, $b < d < 0$, so the length of $\mathbf{s}_1(t)$ decays more quickly than the length of $\mathbf{s}_2(t)$. This leads to an $\boldsymbol{a}(t)$ with larger length (and hence larger energy) than $\boldsymbol{a}(0)$, as sketched in the right panel of Fig. 1(a); thus, transient growth has occurred. For longer times, the length of $\mathbf{s}_2(t)$ also decreases substantially, and the system asymptotically approaches the laminar state with $a_2 = a_3 = 0$.

For other initial conditions, transient energy growth might not occur: see Fig. 1(b). Clearly, the energy initially decreases with time. Depending on the rate of decay of the length of $\mathbf{s}_2(t)$, the energy might always remain below its initial value, or might eventually grow above its initial value. Such considerations motivate the following exploration of how transient energy growth depends on initial conditions.

2.2. Application to the streak-streamwise vortex interaction

A general linear equation $\dot{\boldsymbol{a}} = A\boldsymbol{a}$ has the exact solution $\boldsymbol{a}(t) = e^{tA}\boldsymbol{a}_0$, where $\boldsymbol{a}_0 = \boldsymbol{a}(0)$. The energy of a solution is defined as $E(t) = |\boldsymbol{a}(t)|^2 = \sum a_i^2$, where the sum is over the components of $\boldsymbol{a}$. It is found that $E'(0) = 2\boldsymbol{a}_0 \cdot A\boldsymbol{a}_0 \equiv f(\boldsymbol{a}_0)$. To find the (normalized) initial condition which gives the maximum initial energy growth, $f(\boldsymbol{a}_0)$ is maximized subject to the constraint $g(\boldsymbol{a}_0) \equiv |\boldsymbol{a}_0|^2 = 1$. Using a Lagrange multiplier λ to