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Quantum Games in ion traps

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Abstract

We propose a general, scalable framework for implementing two-choices-multiplayer Quantum Games in ion traps. In particular, we discuss two famous examples: the Quantum Prisoners' Dilemma and the Quantum Minority Game. An analysis of decoherence due to intensity fluctuations in the applied laser fields is also provided.

Keywords: Quantum Games; Ion trap; Decoherence

1. Introduction

Over the last decade the steadily-growing interest in the theory of Quantum Computation and Quantum Information [1] has lead to the development of a number of related topics, one of which is the quantum analogue of Game Theory. Quantum Games (QGs) were introduced in 1999 in the two seminal works of Meyer [2] and Eisert et al. [3] and soon afterwards, quite a number of papers approached the subject ([4–8], surveys are provided in [9] and [10]). The reason why QGs have become so popular is due to their relation with quantum algorithms, especially the oracle-type ones and also with quantum communication and cryptography. Moreover, they provide the opportunity of advancing some fascinating speculations [3,11].

Since QGs are in fact small quantum algorithms, finding efficient ways of implementing them in different physical systems is important for the development of improved quantum computation schemes. Up to the moment, a NMR implementation of the Quantum Prisoners' Dilemma [3] has been experimentally realized [12]. Moreover, a linear optics implementation proposal has been made in [13,14], while recently, a combination of quantum circuit and cluster state model has been suggested [15]. One of the most promising environments for realizing quantum computation is believed to be the ion trap, and therefore, by proposing an ion trap implementation scheme for QGs we fill an important gap in the list of possible physical realizations of QGs. Furthermore, we suggest that the analysis of an ion-trap-specific type of decoherence as a physical process is a very good way of confirming the theoretical results concerning decoherence in QGs and better understanding the physical phenomenon.

This Letter is structured as following: after reviewing the basics of QGs, we introduce a general framework for realizing QGs in the ion trap. We also analyze the effects of a particular type of decoherence that is significant to the proposed scheme and compare the results with those derived by previous researchers on a purely theoretical basis.

2. Quantum Games

The quantization of classical games does not only mean replacing the players' choices with qubits and using unitary transformations on single qubits, but also exploiting entanglement. These lead to new equilibria and increased payoffs for particular quantum strategies. Let us consider the general setup of two-choices-multiplayer games ($2 \times N$). Each player is given a qubit (basis states $|g\rangle$ encoding, for example, the choice of cooperation and $|e\rangle$, say defection). The players' qubits are first entangled (entanglement operator $\hat{J}_N$) and then the players can

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make their choices, that is, apply on their qubits unitary transformations of the following form:

$$\hat{U}(\theta,\phi)=\begin{pmatrix} e^{i\phi}\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix}. \tag{1}$$

For example, the choice of cooperation might be represented by $C=\hat{U}(0,0)$ while the unitary transformation $D=\hat{U}(\pi,0)$ might stand for defection. More generally, the set of strategies can be parametrized by three parameters instead of two, $\hat{V}(\theta,\alpha,\beta)=\begin{pmatrix} e^{i\alpha}\cos\frac{\theta}{2} & ie^{i\beta}\sin\frac{\theta}{2} \\ ie^{-i\beta}\sin\frac{\theta}{2} & e^{-i\alpha}\cos\frac{\theta}{2} \end{pmatrix}$, as we will see later. Before the final measurement, a disentanglement operator ($\hat{J}_N^\dagger$) is applied. This sequence of operations can be written as a quantum circuit (Fig. 1) in which $\hat{U}_1,\ldots,\hat{U}_N$ are the players' moves and $\hat{J}_N$ is the entanglement operator of N qubits. The measured ideal final state is

$$|\psi_f\rangle=\hat{J}_N^\dagger(\hat{U}_1\otimes\hat{U}_2\otimes\cdots\otimes\hat{U}_N)\hat{J}_N|gg\cdots g\rangle. \tag{2}$$

Finally, we need a payoff function associated to all possible measurement results. We have now defined all the necessary components $2\times N$ QGs require.

For a more intuitive understanding, let us discuss two famous examples, the *Quantum Prisoners' Dilemma* (QPD) and *Quantum Minority Game*, whose classical counterparts have numerous applications in economics, social sciences, biology and so on. The QPD is a non-zero sum game in which the Nash equilibrium does not lead to Paretto optimality, meaning that the rational choice of strategies does not provide optimal payoff. Two players (prisoners) must choose between cooperation and defection. Logical reasoning leads both players to defection, which, unfortunately for them, is less efficient than mutual cooperation. The quantized version for two players was first introduced in [3]. Here we consider the entanglement operator $\hat{J}$ of the same form as in [12]:

$$\hat{J}|gg\rangle=\cos\left(\frac{\gamma}{2}\right)|gg\rangle+i\sin\left(\frac{\gamma}{2}\right)|ee\rangle, \tag{3}$$

where $\gamma\in[0,\pi/2]$ is the entanglement strength. In order to have the classical Prisoners' Dilemma included in its quantum version it is necessary that the entanglement operator commutes with any direct product of "classical" strategies C and D defined as before. The payoff function for the first player is $\$_1=rP_{gg}+pP_{ee}+tP_{eg}+sP_{ge}$, where P_{ij}, $i,j\in\{e,g\}$ are the probabilities of measuring the respective $|ij\rangle$, $i,j\in\{e,g\}$ states and $r=3$, reward, $p=1$, punishment, $t=5$, temptation and $s=0$, sucker's payoff, are the values in the payoff matrix of the classical dilemma.

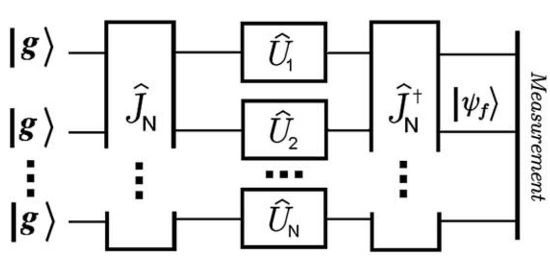

Fig. 1. The quantum circuit for $2\times N$ QGs.

The quantization of the Prisoners' Dilemma brings a whole new set of solutions depending on the degree of entanglement γ. New strategies and new equilibria occur and the classical dilemma is removed in the case of maximal entanglement by using the pure quantum strategy $\hat{Q}=\hat{U}(0,\pi/2)$ for which we obtain both Nash equilibrium and Paretto optimality. These properties have been extensively discussed elsewhere ([3,4,12]) so we will not insist on this aspect.

An example of multiplayer QG is the *Quantum Minority Game* (QMG) [7,16]. N players select one of the two available choices ($\hat{A}=\hat{V}(0,0,0)$ and $\hat{B}=\hat{V}(\pi,0,0)$) and the minority is rewarded. If there is no minority, then nobody is rewarded. Classically, the equilibrium is trivial in the sense that there is no better strategy than making a random choice. Quantumly, for N odd, the quantization brings nothing new but in the case of N even, new Nash equilibria appear and the players' payoffs are increased. For example, for $N=4$, the classical expected payoff is $\frac{1}{8}$, while in the quantum version the expected payoff is $\frac{1}{4}$ and the Nash equilibrium strategy is in this case $\hat{V}(\frac{\pi}{2},-\frac{\pi}{16},\frac{\pi}{16})$. Moreover, in the QMG it is not necessary to apply the disentanglement operator $\hat{J}^\dagger$ before measurement.

3. Implementation scheme

The ion trap has been extensively studied as a candidate system for realizing an efficient quantum information processor. In this work we will refer to the computation schemes in [17,18]. The usual representation of qubits in the ion trap uses the ground ($|g\rangle$) and excited ($|e\rangle$) internal energy states of each ion. Unitary transformations of the form $\hat{U}(\theta,\phi)$ are realized straightforward by using pulses or combinations of laser pulses exciting the carrier transition that couples only the internal states leaving the vibrational mode unchanged. For example, a very simple game like the PQ-coin flip in [2] where two players take turns in flipping a quantum coin (i.e. qubit) would be realized by a sequence of six carrier pulses. The Hamiltonian describing the carrier transition is:

$$H_{\text{carrier}}=\hbar\Omega\left(1-\eta^2a^\dagger a\right)\left[\sigma_+e^{i\phi}+\text{h.c.}\right], \tag{4}$$

where Ω is the Rabi frequency of the transition between the internal states, σ_+ and σ_- are the two-level atom transition operators, η is the Lamb–Dicke parameter, ϕ is the laser phase and $a^\dagger$ and a are the creation and annihilation operators of the vibrational states. It follows that a $k\pi$ carrier pulse ($t=k\pi/\Omega$) realizes one-qubit rotations of the following form:

$$R(\theta,\phi)=\begin{pmatrix} \cos(\frac{|\Omega|t}{2}) & -ie^{i\phi}\sin(\frac{|\Omega|t}{2}) \\ -ie^{-i\phi}\sin(\frac{|\Omega|t}{2}) & \cos(\frac{|\Omega|t}{2}) \end{pmatrix}. \tag{5}$$

However, in general, QGs require entanglement and in some cases, like the QPD, not any entanglement operator can be utilized as it must commute with any direct product of "classical" strategies. In fact, we would rather like to have an entanglement operator of the form $\hat{J}|gg\rangle=\cos(\frac{\gamma}{2})|gg\rangle+i\sin(\frac{\gamma}{2})|ee\rangle$, with $\gamma\in[0,\pi/2]$. Furthermore, we need a general operator that can produce multi-particle entanglement in the same way it pro-