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Solitary wave solutions for a generalized Hirota–Satsuma coupled KdV equation by homotopy perturbation method

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Abstract

In this Letter, He's homotopy perturbation method (HPM), which does not need small parameter in the equation is implemented for solving the nonlinear Hirota–Satsuma coupled KdV partial differential equation. In this method, a homotopy is introduced to be constructed for the equation. The initial approximations can be freely chosen with possible unknown constants which can be determined by imposing the boundary and initial conditions. Comparison of the results with those of Adomian's decomposition method has led us to significant consequences. The results reveal that the HPM is very effective, convenient and quite accurate to systems of nonlinear equations. It is predicted that the HPM can be found widely applicable in engineering.

Keywords: Homotopy perturbation method; Helmholtz equation; FKdV equation; Nonlinear partial differential equations

1. Introduction

In this Letter, we consider a generalized Hirota–Satsuma coupled KdV equation which was introduced by Wu et al. [1]. In [1], the authors introduced a 4×4 matrix spectral problem with three potentials and proposed a corresponding hierarchy of nonlinear equations. One of the typical equations in the hierarchy is a new generalized Hirota–Satsuma coupled KdV equation as follows:

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x, \\ v_t &= -v_{xxx} + 3uv_x, \\ w_t &= -w_{xxx} + 3uw_x. \end{aligned} \tag{1}$$

Eq. (1) is reduced to a new complex coupled KdV equation [1] and the Hirota–Satsuma equation [2] with $w = v^*$ and $w = v$, respectively. More recently, the soliton solutions for this equation were constructed by Fan [3]. In this work, the author has set up two kinds of soliton solutions using an extended tanh-function method and symbolic computation [4,5]. The main idea of this method is to take full advantage of a Riccati equation involving a parameter and to use its solutions to replace the tanh-function in the tanh-function method.

Solitary solutions for various nonlinear wave equations are searched for via various different methods which can only solve a special kind of nonlinear problem due to the limitation or shortcoming in the methods. The discussed generalized Hirota–Satsuma coupled KdV equation has been studied by many authors via different approaches, for example, Jacobi elliptic function method by Yu et al. [6], the projective Riccati equations method by Yong et al. [7], the algebraic method by Zayed et al. [8], variational iteration method by He et al. [9] and Adomian's decomposition method by Kaya [10].

The homotopy perturbation method (HPM) was first proposed by He [11–15]. The HPM does not depend on a small parameter in the equation. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$ which is considered as a "small parameter".

The HPM was successfully applied to nonlinear oscillators with discontinuities [11] and bifurcation of nonlinear problems [14]. In [12], a comparison of HPM and homotopy analysis method was made, revealing that the former is more power-

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ful than the latter. The HPM was proposed to search for limit cycles or bifurcation curves of nonlinear equations [16]. In [13], an heuristical example was given to illustrate the basic idea of the homotopy perturbation method and its advantages over the δ-method, and also this method was applied to solve boundary value problems [17] and heat radiation equations [18].

When implementing the HPM, we get the explicit solutions of the coupled KdV equation without using any transformation method. The method presented here is also simple to use for obtaining numerical solution of the equations without using any discretization techniques. Furthermore, we will show that considerably better approximations related to the accuracy level would be obtained.

2. Basic idea of He's homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation [19]:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2}$$

with the boundary conditions of

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{3}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

Generally speaking, the operator A can be divided into two parts which are L and N, where L is linear, but N is nonlinear. Eq. (2) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{4}$$

By the homotopy technique, we construct a homotopy $V(r,p): \Omega \times [0,1] \to \mathbb{R}$ which satisfies:

$$H(V,p) = (1-p)\big[L(V) - L(u_0)\big] + p\big[A(V) - f(r)\big] = 0, \quad p \in [0,1],\ r \in \Omega, \tag{5}$$

or

$$H(V,p) = L(V) - L(u_0) + pL(u_0) + p\big[N(V) - f(r)\big] = 0, \tag{6}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (2), which satisfies the boundary conditions. Obviously, from Eqs. (5) and (6) we will have:

$$H(V,0) = L(V) - L(u_0) = 0, \tag{7}$$

$$H(V,1) = A(V) - f(r) = 0, \tag{8}$$

the changing process of p from zero to unity is just that of $V(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(V) - L(u_0)$ and $A(V) - f(r)$ are called homotopy.

According to the HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solution of Eqs. (5) and (6) can be written as a power series in p:

$$V = V_0 + pV_1 + p^2V_2 + \cdots. \tag{9}$$

Setting $p = 1$ results in the approximate solution of Eq. (2):

$$u = \lim_{p\to 1} V = V_0 + V_1 + V_2 + \cdots. \tag{10}$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques.

The series (10) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(V)$ (the following opinions are suggested by He [19]):

(1) The second derivative of $N(V)$ with respect to V must be small because the parameter may be relatively large, i.e., $p \to 1$.
(2) The norm of $L^{-1}\partial N/\partial V$ must be smaller than one so that the series converges.

3. Analysis of the method

To investigate the traveling wave solution of Eq. (1), we first construct a homotopy as follows:

$$\begin{aligned}
&(1-p)(\dot v_1 - \dot u_0) + p\left(\dot v_1 - \frac{1}{2}v_1''' + 3v_1v_1' - 3(v_2v_3)'\right) = 0,\\
&(1-p)(\dot v_2 - \dot v_0) + p(\dot v_2 + v_2''' - 3v_1v_2') = 0,\\
&(1-p)(\dot v_3 - \dot w_0) + p(\dot v_3 + v_3''' - 3v_1v_3') = 0,
\end{aligned} \tag{11}$$

where "primes" denote differentiation with respect to x, and "dot" denotes differentiation with respect to t, and the initial approximations are as follows:

$$\begin{aligned}
&v_{1,0}(x,t) = u_0(x,t) = u(x,0),\\
&v_{2,0}(x,t) = v_0(x,t) = v(x,0),\\
&v_{3,0}(x,t) = w_0(x,t) = w(x,0),
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&v_1 = v_{1,0} + pv_{1,1} + p^2v_{1,2} + p^3v_{1,3} + \cdots,\\
&v_2 = v_{2,0} + pv_{2,1} + p^2v_{2,2} + p^3v_{2,3} + \cdots,\\
&v_3 = v_{3,0} + pv_{3,1} + p^2v_{3,2} + p^3v_{3,3} + \cdots,
\end{aligned} \tag{13}$$

where $v_{i,j}$, $i, j = 1, 2, 3, \ldots$ are functions yet to be determined. Substituting Eqs. (9) and (10) into Eq. (8) and arranging the coefficients of "p" powers, we have

$$\begin{aligned}
&\left(\dot v_{1,1} - \frac{1}{2}v_{1,0}''' + \dot v_{1,0} + 3v_{1,0}v_{1,0}' - 3v_{2,0}v_{3,0}' - 3v_{3,0}v_{2,0}'\right)p\\
&\quad + \left(\dot v_{1,2} - \frac{1}{2}v_{1,1}''' - 3v_{2,0}v_{3,1}' + 3v_{1,0}v_{1,1}' - 3v_{2,1}v_{3,0}'\right.\\
&\quad \left. - 3v_{3,1}v_{2,0}' + 3v_{1,1}v_{1,0}' - 3v_{3,0}v_{2,1}'\right)p^2\\
&\quad + \left(\dot v_{1,3} - \frac{1}{2}v_{1,2}''' + 3v_{1,2}v_{1,0}' - 3v_{2,1}v_{3,1}' - 3v_{2,2}v_{3,0}'\right.\\
&\quad - 3v_{3,2}v_{2,0}' - 3v_{3,1}v_{2,1}' + 3v_{1,0}v_{1,2}' - 3v_{3,0}v_{2,2}'\\
&\quad \left. + 3v_{1,1}v_{1,1}' - 3v_{2,0}v_{3,2}'\right)p^3 + \cdots = 0,\\
&(\dot v_{2,1} + v_{2,0}''' + \dot v_{2,0} - 3v_{1,0}v_{2,0}')p\\
&\quad + (\dot v_{2,2} + v_{2,1}''' - 3v_{1,0}v_{2,1}' - 3v_{1,1}v_{2,0}')p^2
\end{aligned}$$