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# Approximation solution of the Dirac equation with position-dependent mass for the generalized Hulthén potential ☆

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**Abstract**

The one-dimensional Dirac equation with position-dependent mass is approximately solved for the generalized Hulthén potential in the case of the smooth step mass distribution. The relativistic energy spectrum and two-component spinor wavefunctions are obtained by the function analysis method. Some interesting results including the standard one-dimensional Hulthén and Woods–Saxon potentials are also discussed.


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*Keywords:* Dirac equation; Generalized Hulthén potential; Energy spectrum

## 1. Introduction

Searching for the exactly solvable solutions of the Schrödinger equation and relativistic wave equation for the real physical potentials have attracted much attention in quantum mechanics. Recently, Egrifes et al. [1,2] investigated the bound states of the Klein–Gordon and Dirac equations for the generalized Hulthén potential [3] by using the Nikiforov–Uvarov method [4] within the framework of PT-symmetric quantum mechanics [5]. In recent years, there has been a growing interest in non-Hermitian Hamiltonian systems with a real energy spectrum [6–11]. The generalized Hulthén potential includes the standard Hulthén potential [12] and Woods–Saxon potential [13] as special cases.

Systems with position-dependent mass have been found to be very useful in studying the physical properties of various microstructures, such as quantum dots [14], semiconductor heterostructure [15], quantum liquids [16], etc. In the last few years, there has been considerable work on searching for analytic solutions of the Schrödinger equation with position-dependent mass [17–28]. In Ref. [23], Alhaidari obtained the exact solution of the Dirac equation for a charged particle with position-dependent mass in the Coulomb field. Considering that there are few work on searching for analytic solutions of the relativistic wave equation with position-dependent mass, thus, searching further for analytic solutions of the relativistic wave equation with position-dependent mass is considerable interest.

In this Letter, we consider the generalized Hulthén potential and the smooth step mass distribution to solve the Dirac equation with position-dependent mass. The relativistic energy spectrum formula and corresponding spinor wavefunction expression for the generalized Hulthén potential have been obtained analytically.

## 2. Bound states of the Dirac equation with position-dependent mass

Choosing the atomic units $\hbar = c = 1$, the one-dimensional time-independent Dirac equation with any given interaction potential $V(x)$ in the vector coupling scheme is given by [29,30]

$$\left[i\frac{d}{dx}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} + \left(E - V(x)\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - M\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\Psi(x) = 0, \quad (1)$$

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where $E$ denotes the energy and $M$ denotes the mass. The spinor wavefunction $\Psi(x)$ has two components. We denote the upper and lower components by $\phi(x)$ and $\theta(x)$, respectively. Eq. (1) can be decomposed into the following two coupled differential equations:

$$-i\frac{d\theta}{dx}+\big[E-V(x)\big]\theta-M(x)\phi=0, \tag{2}$$

$$i\frac{d\phi}{dx}+\big[E-V(x)\big]\phi-M(x)\theta=0. \tag{3}$$

Eliminating the lower spinor component form Eqs. (2) and (3), we obtain a second order differential equation, which contains first order derivatives,

$$-\frac{d^2\phi}{dx^2}+\bigg[2EV(x)-V^2(x)-i\frac{dV(x)}{dx} -i\frac{1}{M(x)}\frac{dM(x)}{dx}\big(E-V(x)\big)\bigg]\phi +\frac{1}{M(x)}\frac{dM(x)}{dx}\frac{d\phi}{dx}=\big[E^2-M^2(x)\big]\phi. \tag{4}$$

We consider the smooth step mass

$$M(x)=M_0(1+\eta\tanh_q\alpha x), \tag{5}$$

where $\eta$ is a very small non-negative parameter, and $q$ is a constant, $q\neq 0$. Here, we have applied deformed hyperbolic functions, which are defined as [31],

$$\sinh_q x=\frac{e^x-qe^{-x}}{2},\qquad \cosh_q x=\frac{e^x+qe^{-x}}{2},$$
$$\operatorname{sech}_q x=\frac{1}{\cosh_q x},\qquad \operatorname{cosech}_q x=\frac{1}{\sinh_q x},$$
$$\tanh_q x=\frac{\sinh_q x}{\cosh_q x},\qquad \coth_q x=\frac{\cosh_q x}{\sinh_q x}. \tag{6}$$

Recently, de Souza Dutra [32] found that the deformed hyperbolic functions can be reduced to the non-deformed ordinary hyperbolic functions potentials by using the coordinate translation transformation. When $q=1$, the mass increases from the value $M=M_0(1-\eta)$ for $x=-\infty$ to the value $M=M_0(1+\eta)$ for $x=+\infty$. The significant variations are occurring in the range of $-\frac{1}{\alpha}<x<\frac{1}{\alpha}$, i.e., $M(-1/\alpha)\cong M_0(1-0.762\eta)$, $M(1/\alpha)\cong M_0(1+0.762\eta)$. The smooth step mass with $q=1$ has been studied by Dekar et al. [17] for a smooth potential within the framework of the one-dimensional Schrödinger equation with position-dependent mass. The ratio of the derivative of the mass to the mass is given by

$$\frac{dM(x)/dx}{M(x)}=\frac{\alpha\eta(1-\tanh_q^2\alpha x)}{1+\eta\tanh_q\alpha x}. \tag{7}$$

In the case of $\eta\to 0$, we may ignore the terms that contain the derivative of the mass in Eq. (4). This approximation can lead us to reduce the Dirac equation (4) to the Klein–Gordon form

$$-\frac{d^2\phi}{dr^2}+\bigg[2EV(x)-V^2(x)-i\frac{dV(x)}{dx}\bigg]\phi =\big[E^2-M^2(x)\big]\phi. \tag{8}$$

We consider the one-dimensional generalized Hulthén potential

$$V(x)=\frac{V_0}{e^{2\alpha x}+q}=\frac{V_0}{2q}(1-\tanh_q\alpha x). \tag{9}$$

After making the parameter replacements of $\alpha\to\alpha/2$, $q\to -q$ and $V_0\to -V_0$, the generalized Hulthén potential (9) becomes the form given by Egrifes et al. in Eq. (11) of Ref. [2]. When $q=-1$ and $q=1$, this generalized Hulthén potential (9) can be reduced to the standard one-dimensional Hulthén potential [12] and Woods–Saxon potential [13], respectively. Substituting Eqs. (5) and (9) into Eq. (8), we can obtain a Schrödinger-like equation for the upper spinor component $\phi(x)$,

$$\bigg[-\frac{d^2}{dx^2}-V_1\operatorname{sech}_q^2\alpha x-V_2\tanh_q\alpha x\bigg]\phi(x)=\tilde{E}^2\phi(x), \tag{10}$$

where the parameters $V_1$, $V_2$ and $\tilde{E}$ are defined as the combinations of some parameters,

$$V_1=-\frac{i}{2}\alpha V_0-\frac{V_0^2}{4q}+q\eta^2M_0^2, \tag{11a}$$

$$V_2=\frac{EV_0}{q}-\frac{V_0^2}{2q^2}-2\eta M_0^2, \tag{11b}$$

$$\tilde{E}^2=E^2-\big(1+\eta^2\big)M_0^2-\frac{EV_0}{q}+\frac{V_0^2}{2q^2}. \tag{11c}$$

We let $z=\frac{1+\tanh_q\alpha x}{2}$ and write the upper spinor component wavefunction as $\phi(x)=z^{-p}(1-z)^{-w}F(z)$, then Eq. (10) can be reduced to the following form

$$z(1-z)\frac{d^2F(z)}{dz^2}+\big[-2p+1-(-2p-2w+1+1)z\big]\frac{dF(z)}{dz} -\bigg[(p+w)^2-p-w-\frac{V_1}{q\alpha^2}-\frac{(\frac{\tilde{E}}{2\alpha})^2+p^2-\frac{V_2}{4\alpha^2}}{z(1-z)} -\frac{-p^2+w^2+\frac{V_2}{2\alpha^2}}{1-z}\bigg]F(z)=0. \tag{12}$$

If and only if the following two equations exist:

$$\left(\frac{\tilde{E}}{2\alpha}\right)^2+p^2-\frac{V_2}{4\alpha^2}=0, \tag{13a}$$

$$-p^2+w^2+\frac{V_2}{2\alpha^2}=0, \tag{13b}$$

Eq. (12) can be reduced to a Gauss hypergeometric equation. A solution of Eq. (12) can be written in Gauss hypergeometric form as

$$\phi(x)=\left(\frac{1+\tanh_q\alpha x}{2}\right)^{-p}\left(\frac{1-\tanh_q\alpha x}{2}\right)^{-w} \times{}_2F_1\bigg(-p-w+\frac{1}{2}-\sqrt{\frac{V_1}{q\alpha^2}+\frac{1}{4}},-p-w+\frac{1}{2} +\sqrt{\frac{V_1}{q\alpha^2}+\frac{1}{4}};-2p+1;\frac{1+\tanh_q\alpha x}{2}\bigg). \tag{14}$$