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Classes of exact Klein–Gordon equations with spatially dependent masses: Regularizing the one-dimensional inversely linear potential

A. de Souza Dutra [a,*], Chun-Sheng Jia [b]

[a] UNESP-Campus de Guaratinguetá-DFQ, Av. Dr. Ariberto Pereira da Cunha, 333, C.P. 205, 12516-410 Guaratinguetá, SP, Brazil
[b] State Key Laboratory of Oil and Gas Reservoir Geology and Exploitation, Southwest Petroleum University, Chengdu 610500, People's Republic of China

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Abstract

In this work we consider the effect of a spatially dependent mass over the solution of the Klein–Gordon equation in $1+1$ dimensions, particularly the case of inversely linear scalar potential, which usually presents problems of divergence of the ground-state wave function at the origin, and possible nonexistence of the even-parity wave functions. Here we study this problem, showing that for a certain dependence of the mass with respect to the coordinate, this problem disappears.

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The problem of the nonrelativistic one-dimensional Coulomb potential has been attracting some attention in the recent literature [1,2]. In particular, very recently, de Castro [2] has discussed the problem from the point of view of the relativistic bosonic particles, governed by a Klein–Gordon equation with mixed vector–scalar inversely linear potentials. In fact, the problem of exact solutions of the Klein–Gordon equation for a number of special potentials, has also been a line of great interest in the recent years [3–5]. On the other hand the one-dimensional inversely linear scalar potential shares some of the problems of the one-dimensional hydrogen atom, which presents some intriguing features like the possible nonexistence of even-parity eigensolutions. This, and the possible quite unusual ground-state given by a Dirac delta function with infinity energy as claimed by Loudon [6], are some of the principal reasons why it is a so attractive problem. In [2], de Castro concluded in favor of the nonexistence of the even-parity solutions of such system, by taking the nonrelativistic limit of his results, and so reinforcing the idea present in some papers where the problem was considered directly at the nonrelativistic regime [1].

This kind of system can be thought physically as a quantum wire, where the particle is restricted to move basically along one line, and furthermore there are inversely linear interactions, besides the existence of an effective spatially dependent mass.

On the other hand, the problem of the spatially dependent mass is presenting a growing interest along the last few years [7–25]. It is quite natural to look for relativistic treatment of this type of systems, mostly because the ordering ambiguity which is present in the nonrelativistic case [7], is expected to be avoided under relativistic ambiance [9–11].

However, at this point it is important to remark that there are great difficulties to define from first principles the Dirac equation for fermions or, alternatively the Klein–Gordon one for bosons, when one have in mind to consider space–time dependent masses. For the case of free fermions, it is not difficult to verify that the Dirac equation

$$i\gamma^{\mu}\partial_{\mu}\psi = m\psi, \quad \mu = 0, 1, 2, 3, \tag{1}$$

where relativistic units $\hbar = c = 1$ are being used, can be squared to

$$\partial^{\mu}\partial_{\mu}\psi = -m^2\psi, \tag{2}$$

* Corresponding author.
E-mail addresses: dutra@feg.unesp.br (A. de Souza Dutra), chshjia@263.net (C.-S. Jia).

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showing the compatibility with the Klein–Gordon equation. However, when one takes into account for instance a spatial dependence of the mass, the final equation looks like

$$\partial^\mu \partial_\mu \psi = -\left[m^2 + i\gamma^\mu \partial_\mu m\right]\psi. \tag{3}$$

So, presenting a residual spin contribution. On the other hand, there is the fact that the physical particles in quantum field theory must belong to an irreducible representation of the Poincaré algebra [12,13]. In other words, it is necessary in general that the particle be specified through the Casimir invariants $P^\mu P_\mu = m^2$ and $J^\mu P_\mu = \alpha m$, with m and α standing respectively for the mass and helicity of the particle. It is quite difficult to construct the generators and Casimir operators of the Poincaré algebra when one has a position-dependent mass. As a consequence, one must have in mind that what is usually thought as relativistic equations for position dependent masses, should be considered as effective equations. This is the spirit which we are going to develop through this work for the case of effective bosonic particles under effective relativistic Klein–Gordon equations.

The idea here is to show that the problem of the one-dimensional inversely linear scalar potential could be circumvented if there exist a position-dependent mass of a specific form. So, in a certain way, the spatial dependence of the mass can regularize the system, avoiding nonphysical divergences and restoring the even-parity wave functions. Let us try to present the question by rewriting the time-independent Klein–Gordon equation for a spatially dependent bosonic mass in $1+1$ dimensions

$$-\hbar^2 c^2 \frac{d^2\psi}{dx^2} + \left(m(x)c^2 + S(x)\right)^2 \psi = \left(E - V(x)\right)^2 \psi, \tag{4}$$

into a more familiar one, usually used in nonrelativistic systems, the Schrödinger equation. In doing so, one gets an equation given by

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + V_{\text{eff}}(x)\psi(x) = \mathcal{E}\psi(x), \tag{5}$$

where $\mathcal{E} \equiv \frac{E^2}{2c^2}$, and

$$V_{\text{eff}}(x) \equiv \frac{1}{2c^2}\left(S(x)^2 - V(x)^2 + m(x)^2 c^4\right) + \left(\frac{EV(x)}{c^2} + m(x)S(x)\right). \tag{6}$$

In his paper [2], de Castro used the vector and the scalar potentials inversely proportional to the absolute value of the coordinate. Here we are going to use $V(x) = 0$; $S(x) = \frac{s}{|x|}$ and $m(x) = \frac{m_0}{L}|x|$, with L being a constant with space dimension. So obtaining

$$V_{\text{eff}}(x) = \frac{s^2}{2c^2}\frac{1}{x^2} + \frac{m_0^2 c^2}{2L^2}x^2 + \frac{m_0 s}{L}. \tag{7}$$

So, we finish with an equation of the Schrödinger kind, given by

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \left(\frac{s^2}{2c^2}\frac{1}{x^2} + \frac{m_0^2 c^2}{2L^2}x^2\right)\psi(x) = \left(\mathcal{E} - \frac{m_0 s}{L}\right)\psi(x), \tag{8}$$

where the final potential is that of a harmonic oscillator with centrifugal barrier, so avoiding the term proportional to $1/|x|$, in the effective nonrelativistic potential, which is the mathematical cause for the illness of the usual one-dimensional hydrogen atom [1,2]. At this point we gathered conditions to get the exact energy and the wave functions of this problem in an easy way. For instance, one can use the fact that this is a shape invariant potential [26] and use the supersymmetric quantum mechanics in order to achieve the energy levels and eigenfunctions [27–32]. This can be done multiplying the above equation by $2/\hbar^2$ and defining the ground-state wave function in terms of the so-called superpotential $W(x)$, so that it can be written in the form

$$\psi_0(x) = \exp\left(-\int W(x)\,dx\right), \tag{9}$$

from which we get the first-order nonlinear differential equation governing the superpotential,

$$W^2(x) - \frac{dW(x)}{dx} = \left(\frac{s^2}{\hbar^2 c^2}\frac{1}{x^2} + \frac{m_0^2 c^2}{\hbar^2 L^2}x^2\right) - \tilde{E}_0, \tag{10}$$

where $\tilde{E} \equiv \frac{2}{\hbar^2}(\frac{E^2}{2c^2} - \frac{m_0 s}{L})$. Note that the above equation is the well-known nonlinear Riccati equation. By choosing $W(x)$ as

$$W(x) = \frac{A}{x} + Bx. \tag{11}$$

In order to be a solution of the Riccati equation, one needs to impose some restrictions over the ansatz parameters as

$$A^2 + A = \frac{s^2}{\hbar^2 c^2}, \qquad B^2 = \frac{m_0^2 c^2}{\hbar^2 L^2}, \qquad 2AB - B = -\tilde{E}_0. \tag{12}$$

Furthermore, the ground-state wave function is written in this case as

$$\psi_0(x) = x^{-A} e^{-\frac{B}{2}x^2}. \tag{13}$$

In the present work, we deal with bound state solutions, i.e., the wave function $\psi(x)$ must satisfy boundary conditions that makes it becomes zero when $|x| \to \infty$, and $\psi(x) = 0$ at the origin point, $x = 0$. In order to guarantee that the ground-state wave function obeys these boundary conditions, we must impose that $A < 0$ and $B > 0$. Solving Eqs. (12), we obtain that

$$A = -\frac{1}{2}\left(1 + \sqrt{1 + \frac{4s^2}{\hbar^2 c^2}}\right), \qquad B = \frac{m_0 c}{\hbar L}, \qquad \tilde{E}_0 = B(1 - 2A). \tag{14}$$

In terms of the superpotential $W(x)$ given in Eq. (11), we can construct the following two supersymmetric partner potentials