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Homotopy perturbation method for thin film flow of a fourth grade fluid down a vertical cylinder

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Abstract

The present Letter applies the homotopy perturbation method and the traditional perturbation method to obtain analytic approximations of the non-linear equations modeling thin film flow of a fourth grade fluid falling on the outer surface of an infinitely long vertical cylinder. Expressions for the velocity, volume flux and average velocity are obtained. Comparison of the results obtained by the two methods reveal that homotopy perturbation method is more effective and easy to use.

Keywords: Thin film flow; Fourth grade fluid; Perturbation method; Homotopy perturbation method

1. Introduction

Approximate analytical and numerical methods are widely used to solve non-linear differential equations modeling physical phenomena. This is because exact solutions of these equations are rare. Most of the perturbation methods [1–3] require the presence of a small parameter in the equation. Sometimes the small parameter, if not already present in the equation, is artificially introduced and finally set equal to unity to obtain the solution of the original problem.

Recently, He [4–6] proposed a new perturbation method which is, in fact, a coupling of the traditional perturbation method and homotopy as used in topology. This is the homotopy perturbation method (HPM). In his several papers He applied this method to discuss non-linear boundary value problems [7] as well as non-linear problems on bifurcation [8,9], asymptotology [10], wave equation [11] and oscillator with discontinuities [12]. Because of the success of homotopy perturbation method, Abbasbandy [13] used it for Laplace transform, Cveticanin [14] applied it to study pure non-linear differential equations and El-Shahed [15] applied this technique to the integro-differential equation for Volterra's model. Turning to fluid mechanics, Siddiqui et al. [16,17] have invoked this method for solving non-linear problems involving non-Newtonian fluids.

Concerning thin film flow problems, Nuttall [18] used bounding methods to study the thin flow of a viscous incompressible fluid in an inclined uniform channel. In a different direction He first established a variational formulation for nano thin film lubrication [19] by the semi-inverse method [20]. Using different geometries thin flow problems have been studied by Kapitza [21], Yih [22], Krishna and Lin [23], Andersson and Dahi [24] and Po-Jen Cheng [25] and others.

In this Letter, we apply the homotopy perturbation method to discuss the non-linear problem of thin film flow down a vertical cylinder. The fluid used is of fourth grade which introduces more nonlinearities in the problem. It may be remarked that the

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problem was not solved earlier even by traditional perturbation technique. The results obtained in this Letter will now be available for experimental verification to give confidence for the well-posedness of this non-linear boundary value problem.

The Letter is organized as follows. Section 2 contains the governing equations of the fluid model. In Section 3 the problem under consideration is formulated. In Section 4, the governing equation of the problem is solved using perturbation method. In Section 5 the same problem is solved using homotopy perturbation method. In Section 6 concluding remarks are given and the two methods are compared.

2. Basic equations

The basic equations governing the motion of an incompressible fluid, neglecting the thermal effects and body forces, are

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \operatorname{div} \boldsymbol{\tau}, \tag{2}$$

where ρ the constant density, $\mathbf{V}$ the velocity vector, p the pressure, $\boldsymbol{\tau}$ the stress tensor and $\frac{D}{Dt}$ denoting the material derivative.

As discussed in [26], the stress tensor defining a fourth grade fluid is given by

$$\boldsymbol{\tau} = \sum_{i=1}^{4} \mathbf{S}_i, \tag{3}$$

where

$$\begin{aligned}
\mathbf{S}_1 &= \mu \mathbf{A}_1, \\
\mathbf{S}_2 &= \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \\
\mathbf{S}_3 &= \beta_1 \mathbf{A}_3 + \beta_2 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \beta_3 (\operatorname{tr} \mathbf{A}_2) \mathbf{A}_1, \\
\mathbf{S}_4 &= \gamma_1 \mathbf{A}_4 + \gamma_2 (\mathbf{A}_3 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_3) + \gamma_3 \big(\mathbf{A}_2^2\big) + \gamma_4 \big(\mathbf{A}_2 \mathbf{A}_1^2 + \mathbf{A}_1^2 \mathbf{A}_2\big) + \gamma_5 \big((\operatorname{tr} \mathbf{A}_2) \mathbf{A}_2\big) + \gamma_6 \big((\operatorname{tr} \mathbf{A}_2) \mathbf{A}_1^2\big) \\
&\quad + \big(\gamma_7 (\operatorname{tr} \mathbf{A}_3) + \gamma_8 (\operatorname{tr} \mathbf{A}_2 \mathbf{A}_1)\big) \mathbf{A}_1,
\end{aligned}$$

and where μ is the coefficient of viscosity and α_1, α_2, β_1, β_2, β_3, γ_1, γ_2, γ_3, γ_4, γ_5, γ_6, γ_7 and γ_8 are material constants. The Rivilin–Ericksen tensors, $\mathbf{A}_n$, are defined by

$$\mathbf{A}_0 = \mathbf{I}, \quad \text{the identity tensor,}$$

$$\mathbf{A}_n = \frac{D\mathbf{A}_{n-1}}{Dt} + \mathbf{A}_{n-1}(\nabla \mathbf{V}) + (\nabla \mathbf{V})^{\mathrm{T}} \mathbf{A}_{n-1}, \quad n \geqslant 1. \tag{4}$$

3. Formulation of problem

A non-Newtonian fluid of fourth grade falling on the outside surface of an infinitely long vertical cylinder of radius R. The flow is in the form of a thin, uniform axisymmetric film of thickness δ, in contact with stationary air. We shall seek a velocity field of the form

$$\mathbf{v} = \big[0, 0, u(r)\big]. \tag{5}$$

The r-component of momentum is

$$\frac{\partial p}{\partial r} = (2\alpha_1 + \alpha_2) \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\frac{du}{dr} \right)^2 \right] + \frac{4}{r} \left(\gamma_3 + \gamma_4 + \gamma_5 + \frac{\gamma_6}{2} \right) \frac{\partial}{\partial r} \left[r \left(\frac{du}{dr} \right)^4 \right], \tag{6}$$

θ-component of momentum is

$$\frac{\partial p}{\partial \theta} = 0, \tag{7}$$

and z-component of momentum is

$$\frac{\partial p}{\partial z} = \frac{\eta}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) + \frac{2}{r} (\beta_2 + \beta_3) \frac{d}{dr} \left[r \left(\frac{du}{dr} \right)^3 \right]. \tag{8}$$

The continuity equation is satisfied identically and (7) shows that pressure depends only on r and z.