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Exact solutions to sine-Gordon-type equations

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Abstract

In this Letter, sine-Gordon-type equations, including single sine-Gordon equation, double sine-Gordon equation and triple sine-Gordon equation, are systematically solved by Jacobi elliptic function expansion method. It is shown that different transformations for these three sine-Gordon-type equations play different roles in obtaining exact solutions, some transformations may not work for a specific sine-Gordon equation, while work for other sine-Gordon equations.

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1. Introduction

Sine-Gordon-type equations, including single sine-Gordon (SSG for short) equation

$$u_{xt} = \alpha \sin u, \tag{1}$$

double sine-Gordon (DSG for short) equation

$$u_{xt} = \alpha \sin u + \beta \sin 2u, \tag{2}$$

and triple sine-Gordon (TSG for short) equation

$$u_{xt} = \alpha \sin u + \beta \sin 2u + \gamma \sin 3u, \tag{3}$$

are widely applied in physics and engineering. For example, DSG equation is a frequent object of study in numerous physical applications, such as Josephson arrays, ferromagnetic materials, charge density waves, smectic liquid crystal dynamics [1–5]. Actually, SSG equation and DSG equation also arise in nonlinear optics ^{3}He spin waves and other fields. In a resonant fivefold degenerate medium, the propagation and creation of ultra-short optical pulses, the SSG and DSG models are usually used. However, in some cases, one has to consider other sine-Gordon equations. For instance, TSG equation is used to describe the propagation of strictly resonant sharp line optical pulses [6].

Due to the wide applications of sine-Gordon-type equations, many solutions to them have been obtained in different functional forms, such as $\tan^{-1}\coth$, $\tan^{-1}\tanh$, $\tan^{-1}\operatorname{sech}$, $\tan^{-1}\operatorname{sn}$ and so on, by different methods [7–9]. In this Letter, we will show the systematical results about solutions for these three sine-Gordon-type equations. Here different transformations will be introduced to derive more types of solutions, of course, some transformations may not work for a specific sine-Gordon-type equation.

2. Solutions to SSG equation

In order to solve the sine-Gordon-type equations, certain transformations must be introduced. For example, the transformation

$$u = 2\tan^{-1} v \quad \text{or} \quad v = \tan\frac{u}{2}, \tag{4}$$

has been introduced in Refs. [7,9] to solve DSG equation.

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When the transformation (4) is considered, there are

$$\sin u = \frac{2\tan\frac{u}{2}}{1+\tan^2\frac{u}{2}} = \frac{2v}{1+v^2}, \tag{5}$$

and

$$u_{tx} = \frac{2}{1+v^2}v_{tx} - \frac{4v}{(1+v^2)^2}v_t v_x. \tag{6}$$

Combining (5) and (6) with (1), the SSG equation can be rewritten as

$$\left(1+v^2\right)v_{tx} - 2vv_t v_x - \alpha v - \alpha v^3 = 0. \tag{7}$$

Eq. (7) can be solved in the frame

$$v = v(\xi), \quad \xi = k(x-ct), \tag{8}$$

where k and c are wave number and wave speed, respectively.

Substituting (8) into (7), we have

$$k^2c\left(1+v^2\right)\frac{d^2v}{d\xi^2} - 2k^2cv\left(\frac{dv}{d\xi}\right)^2 + \alpha v + \alpha v^3 = 0, \tag{9}$$

which can be solved directly by Jacobi elliptic function expansion method [10,11]. For instance, the ansatz solution can be written as

$$v = a_0 + a_1 \operatorname{sn}\xi, \tag{10}$$

where $\operatorname{sn}\xi$ is Jacobi elliptic sine function [12–14].

The constants a_0 and a_1 can be determined by substituting (10) into (9) as

$$a_0 = 0, \qquad a_1 = \sqrt{-\frac{\alpha-(1+m^2)k^2c}{2k^2c}}, \tag{11}$$

with

$$c = \pm\frac{\alpha}{k^2(1-m^2)}, \tag{12}$$

where $0 \leqslant m \leqslant 1$ is called modulus of Jacobi elliptic functions, see [12–14].

Thus, the solution to the SSG equation is

$$u_{1S} = 2\tan^{-1}\left[\sqrt{-\frac{\alpha-(1+m^2)k^2c}{2k^2c}}\operatorname{sn}\xi\right], \quad m \neq 1. \tag{13}$$

The second transformation is introduced in the form

$$u = 2\sin^{-1}v \quad \text{or} \quad v = \sin\frac{u}{2}, \tag{14}$$

and then

$$\sin u = 2\sin\frac{u}{2}\cos\frac{u}{2} = 2v\sqrt{1-v^2}, \tag{15}$$

and

$$u_{tx} = \frac{2}{\sqrt{1-v^2}}v_{tx} + \frac{2v}{(1-v^2)\sqrt{1-v^2}}v_t v_x. \tag{16}$$

Combining (15) and (16) with (1), the SSG equation can be rewritten as

$$\left(1-v^2\right)v_{tx} + vv_t v_x - \alpha v\left(1-v^2\right)^2 = 0. \tag{17}$$

In the travelling wave frame (8), the formal solution of Eq. (17) by the Jacobi elliptic function expansion method [10, 11] can be written as

$$v = a_0 + a_1\operatorname{sn}\xi. \tag{18}$$

Similarly, the expansion coefficients can be determined as

$$a_0 = 0, \qquad a_1 = \pm 1, \qquad c = \frac{\alpha}{m^2k^2}, \tag{19}$$

or

$$a_0 = 0, \qquad a_1 = \pm m, \qquad c = \frac{\alpha}{k^2}. \tag{20}$$

Thus, we can obtain two more solutions to the SSG equation:

$$u_{2S} = \pm 2\sin^{-1}[\operatorname{sn}\xi], \tag{21}$$

and

$$u_{3S} = \pm 2\sin^{-1}[m\operatorname{sn}\xi]. \tag{22}$$

Moreover, it is known that when $m \to 1$, $\operatorname{sn}(\xi, m) \to \tanh\xi$. So we can get more kinds of solution expressed in terms of hyperbolic function,

$$u_{4S} = \pm 2\sin^{-1}[\tanh\xi], \tag{23}$$

with

$$c = \frac{\alpha}{k^2}. \tag{24}$$

Next, we introduce the third transformation

$$u = \cos^{-1}v \quad \text{or} \quad v = \cos u, \tag{25}$$

and then

$$\sin u = \sqrt{1-v^2}, \tag{26}$$

and

$$u_{tx} = -\frac{1}{\sqrt{1-v^2}}v_{tx} - \frac{v}{(1-v^2)\sqrt{1-v^2}}v_t v_x. \tag{27}$$

Combining (26) and (27) with (1), the SSG equation can be rewritten as

$$\left(1-v^2\right)v_{tx} + vv_t v_x + \alpha\left(1-v^2\right)^2 = 0. \tag{28}$$

In the travelling wave frame (8), the formal solution of Eq. (28) by the Jacobi elliptic function expansion method [10, 11] can be written as

$$v = a_0 + a_1\operatorname{sn}\xi + a_2\operatorname{sn}^2\xi. \tag{29}$$

Similarly, the expansion coefficients can be determined as

$$a_0 = \pm 1, \qquad a_1 = 0, \qquad a_2 = -\frac{2m^2k^2c}{\alpha}, \tag{30}$$

with

$$c = \frac{\alpha}{m^2k^2} \quad \text{for } a_0 = 1, \tag{31}$$

and

$$c = \frac{\alpha}{k^2} \quad \text{for } a_0 = -1, \tag{32}$$