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Harmonic balance method for solving a large-amplitude oscillation of a conservative system with inertia and static non-linearity

Md. Helal Uddin Molla *, Md. Abdur Razzak, M.S. Alam

Department of Mathematics, Rajshahi University of Engineering and Technology (RUET), Kazla, Rajshahi 6204, Bangladesh

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In this paper, an analytical approximate technique based on harmonic balance method (HBM) is presented to obtain the approximate frequencies and the corresponding periodic solutions of a conservative oscillator having inertia and static non-linearity. The results of the present paper are valid for small and large amplitudes of oscillation. In previous articles, the first and second approximations were determined for the same oscillator; but the results were not close to the exact result. On the contrary, the new results of this paper are very close to the exact result.

Introduction

In both science and engineering, there exist many non-linear problems in which parameters are not small. To overcome these shortcomings, many analytical techniques such as variational iterative method [1,2], homotopy perturbation method [3–5], variational approach [6] are used to solve strongly nonlinear equations. Recently, the homotopy perturbation method (HPM) has been modified named as optimal homotopy perturbation method (OHPM) [7–10]. The harmonic balance method (HBM) [11–16] is a widely used technique for solving strongly nonlinear systems. Furthermore, some authors [17,18,5,19–23] have investigated a number of nonlinear oscillator problems using various methods. Moreover, many numerical methods [24–33] have been studied to solve differential equations. Nonlinear of planar, large-amplitude free vibrations of a slender, inextensible cantilever beam carrying a lumped mass with rotary inertia at an intermediate position along its span is one of the problems that does not contain small parameter. These problems are not amenable to exact treatment for their complexity and thus the approximate techniques must be needed to solve such problems [34,35,10]. Hamdan and Dado [34] used the harmonic balance method to solve such problems, but they obtained only the approximate periods of oscillation. Recently, some authors [35,10,36,37] have investigated the same oscillator using different methods. In the previous article, Herisanu and Marinca [10] used the optimal homotopy asymptotic method (OHAM) for solving such nonlinear oscillator, but the solution procedure is laborious; moreover, the solution contains up to seventh harmonic terms. On the other hand, in another article [37], the second-order approximation has determined such a nonlinear oscillator, but the solution procedure is also laborious; furthermore, the results are not significantly better as compared with numerical result. The main aim of this article is to provide a harmonic balance technique which contains only fifth harmonic terms to determine the higher approximate solutions of a non-linear conservative system with inertia and static non-linearity.

The formulation as well as the determination of the present paper is systematic and quite easy. Furthermore, the solution contains only a few harmonic terms (containing up to three to five) and these terms make the solution rapidly congregate. The appeared algebraic equations in this paper are analytically solved. Moreover, the results are better than other existing results [10,37,38].

Formulation and solution method

Let us consider a nonlinear differential equation

$$\ddot{x} + x = -f(x,\dot{x}), \quad x(0) = a,\ \dot{x}(0) = 0 \tag{1}$$

where $f(x,\dot{x})$ is a nonlinear function such that $f(-x,-\dot{x}) = -f(x,\dot{x})$.

A periodic solution of Eq. (1) is obtained in the form

$$x(t) = a(\rho\cos\varphi + u\cos 3\varphi + v\cos 5\varphi + w\cos 7\varphi + \cdots) \tag{2}$$

where a, ρ are constants, $\varphi = \omega t$ and $\omega = \frac{2\pi}{T}$ are a frequency of nonlinear oscillation, here T is a period. If $\rho = 1 - u - v - \cdots$ and the initial phase $\varphi_0 = 0$, solution Eq. (2) readily satisfies the initial conditions $x(0) = a, \dot{x}(0) = 0$.

Substituting Eq. (2) into Eq. (1) and expanding $f(x,\dot{x})$ in a Fourier series, it turns to an algebraic identity

* Corresponding author.

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$$a[\rho(1-\dot{\varphi}^2)\cos\varphi + u(1-9\dot{\varphi}^2)\cos 3\varphi\cdots] = -[F_1(a,u,\ldots)\cos\varphi + F_3(a,u,\ldots)\cos 3\varphi\cdots]. \tag{3}$$

By comparing the coefficients of equal harmonics of Eq. (3), the following nonlinear algebraic equations are found

$$\rho(1-\dot{\varphi}^2) = -F_1, \quad u(1-9\dot{\varphi}^2) = -F_3, \quad v(1-25\dot{\varphi}^2) = -F_5,\ldots \tag{4}$$

With the help of first equation, $\dot{\varphi}$ is eliminated from all the rest of Eq. (4). Thus Eq. (4) takes the following form

$$\rho\dot{\varphi}^2 = \rho + F_1, \quad 8u\rho = \rho F_3 - 9uF_1, \quad 24v\rho = \rho F_5 - 25vF_1,\ldots \tag{5}$$

Substituting $\rho = 1-u-v-\cdots$, and after simplifying Eq. (5) takes the following nonlinear algebraic equations as

$$G_1(a,u,v,\cdots) = 0, \; G_2(a,u,v,\cdots) = 0,\ldots \tag{6}$$

Recently, these types of algebraic equations have been solved by the power series method introducing a small parameter (see [15,16] for details). However, for the large amplitude of oscillation, such power series does not converge. In that case, Eq. (6) is solved by a numerical technique. Fortunately, for large-amplitude oscillation of a conservative system with inertia and static non-linearity these equations are truncated to a quadratic form which provides desired results.

Examples

Example 1

Consider the nonlinear oscillator [10,36,37]

$$\ddot{x} + x + \alpha x^2\ddot{x} + \alpha x\dot{x}^2 + \beta x^3 = 0, \tag{7}$$

subject to the initial conditions

$$x(0) = a, \quad \dot{x}(0) = 0. \tag{8}$$

The second-order approximate solution is chosen in the following form

$$x = a(\rho\cos\varphi + u\cos 3\varphi) \tag{9}$$

where $\rho = 1-u$ and $\varphi = \omega t$.

Substituting Eq. (9) into Eq. (7) and expanding in a Fourier series and equating the coefficient of $\cos\varphi$ and $\cos 3\varphi$, we obtained the following equations as

$$1 - u + \frac{3a^2\beta}{4} - \frac{3}{2}a^2u\beta + \frac{9}{4}a^2u^2\beta - \frac{3}{2}a^2u^3\beta - \omega^2 + u\omega^2 - \frac{1}{2}a^2\alpha\omega^2 - \frac{7}{2}a^2u^2\alpha\omega^2 + 4a^2u^3\alpha\omega^2 = 0, \tag{10}$$

$$u + \frac{a^2\beta}{4} + \frac{3}{4}a^2u\beta - \frac{9}{4}a^2u^2\beta + 2a^2u^3\beta - 9u\omega^2 - \frac{1}{2}a^2\alpha\omega^2 - \frac{7}{2}a^2u\alpha\omega^2 + \frac{17}{2}a^2u^2\alpha\omega^2 - 9a^2u^3\alpha\omega^2 = 0, \tag{11}$$

Now, by eliminating ω^2 from Eqs. (10) and (11) and neglecting the higher order of u more than u^2, we obtain the following equation as

$$(2a^2\alpha - a^2\beta + a^4\alpha\beta + (32 + 12a^2\alpha + 24a^2\beta + 7a^4\alpha\beta)u + (-36a^2\alpha - 18a^2\beta - 34a^4\alpha\beta)u^2 = 0 \tag{12}$$

Solving Eq. (12), we obtain the unknown coefficient, u as

$$u(a) = \frac{32 + 12a^2\alpha + 24a^2\beta + 7a^4\alpha\beta + 7a^4\alpha\beta + \sqrt{\lambda}}{2(36a^2\alpha + 18a^2\beta + 34a^4\alpha\beta)} \tag{13}$$

where

$$\lambda = 1024 + 768a^2\alpha + 432a^4\alpha^2 + 1536a^2\beta + 1024a^4\alpha\beta + 584a^6\alpha^2\beta + 504a^4\beta^2 + 272a^6\alpha\beta^2 + 185a^8\alpha^2\beta^2$$

and the second approximate frequency, ω_2 is obtained from (10) as

$$\omega_2^2 = \omega^2 = \frac{4 + 3a^2(1 - u + 2u^2)\beta}{2(2 + a^2(1 + u + 8u^2)\alpha)} \tag{14}$$

The third-order approximation is of the form:

$$x = a(\rho\cos\varphi + u\cos 3\varphi + v\cos 5\varphi) \tag{15}$$

where $\rho = 1-u-v$ and $\varphi = \omega t$.

Substituting Eq. (15) into Eq. (7) and expanding in a Fourier series and equating the coefficient of $\cos\varphi$, $\cos 3\varphi$ and $\cos 5\varphi$, we obtained the following equations as

$$1 - u - v + \frac{3a^2\beta}{4} - \frac{3}{2}a^2u\beta + \frac{9}{4}a^2u^2\beta - \frac{9}{4}a^2v\beta + \frac{9}{2}a^2uv\beta + \frac{15}{4}a^2v^2\beta - \omega^2 + u\omega^2 + v\omega^2 - \frac{1}{2}a^2\alpha\omega^2 - \frac{7}{2}a^2u^2\alpha\omega^2 + \frac{3}{2}a^2v\alpha\omega^2 - 9a^2uv\alpha\omega^2 - \frac{29}{2}a^2v^2\alpha\omega^2 + \cdots = 0, \tag{16}$$

$$u + \frac{a^2\beta}{4} + \frac{3}{4}a^2u\beta - \frac{9}{4}a^2u^2\beta - \frac{3}{2}a^2uv\beta - \frac{3}{4}a^2v^2\beta - 9u\omega^2 - \frac{1}{2}a^2\alpha\omega^2 - \frac{7}{2}a^2u\alpha\omega^2 + \frac{17}{2}a^2u^2\alpha\omega^2 - 3a^2v\alpha\omega^2 + 5a^2uv\alpha\omega^2 + \frac{15}{2}a^2v^2\alpha\omega^2 + \cdots = 0, \tag{17}$$

$$v + \frac{3}{4}a^2u\beta - \frac{3}{4}a^2u^2\beta + \frac{3}{2}a^2v\beta - \frac{9}{2}a^2uv\beta - 3a^2v^2\beta - 25v\omega^2 - \frac{9}{2}a^2u\alpha\omega^2 + \frac{7}{2}a^2u^2\alpha\omega^2 - 13a^2v\alpha\omega^2 + 35a^2uv\alpha\omega^2 + 26a^2v^2\alpha\omega^2 + \cdots = 0. \tag{18}$$

Now eliminating ω^2 from Eqs. (16), (17) and (18), we obtain the following equations as

$$(2a^2\alpha - a^2\beta + a^4\alpha\beta + (32 + 12a^2\alpha + 24a^2\beta + 7a^4\alpha\beta)u - (36a^2\alpha + 18a^2\beta + 34a^4\alpha\beta)u^2 + (12a^2\alpha + 7a^4\alpha\beta)v - (16a^2\alpha + 48a^2\beta + 43a^4\alpha\beta)uv = 0 \tag{19}$$

$$(18a^2\alpha - 3a^2\beta + 12a^4\alpha\beta)u + (96 + 50a^2\alpha + 69a^2\beta + 36a^4\alpha\beta)v = 0 \tag{20}$$

Solving Eqs. (19) and (20), we get the value of u and v .

Solving Eq. (16), we obtained the third-order approximate frequency ω_3 by truncated higher order (more than second order of u and first order of v) as:

$$\omega_3^2 = \omega^2 = \frac{4 + 3a^2\beta - 3a^2u\beta + 6a^2u^2\beta - 6a^2v\beta + 9a^2uv\beta}{2(2 + a^2\alpha + a^2u\alpha + 8a^2u^2\alpha - 2a^2v\alpha + 17a^2uv\alpha)}, \tag{21}$$

where u and v are given in Eqs. (19), (20).

Therefore, the third-order approximate solution of Eq. (7) becomes

$$x = a((1-u-v)\cos\varphi + u\cos 3\varphi + v\cos 5\varphi), \tag{22}$$

where u and v and ω are found from Eqs. (19)–(21).

Example 2

Consider the nonlinear equation [38] in the following form

$$\ddot{x} + x + \alpha x^4\ddot{x} + 2\alpha\dot{x}^2x^3 + \beta x^5 = 0, \tag{23}$$