Results in Physics 6 (2016) 461–467

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.journals.elsevier.com/results-in-physics

Time evolution approach to steady state

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ARTICLE INFO

Article history:
Received 24 April 2016
Accepted 14 June 2016
Available online 24 June 2016

Keywords:
Time evolution equations
Hydrodynamic variables
Ergodic behavior
Navier–Stokes equation
Recurrence
Boltzmann equation

ABSTRACT

We apply a time evolution approach to the statistical mechanics of one and two dimensional systems to study the evolution toward steady state. We have used the Feynman definition of an inverse operator to show that in one and two dimensions, there is an approach to steady state of hydrodynamic variables such as field velocities and pressure. Illustrative examples in 1D are shown to display steady state variables.

Introduction

In the earliest version of an exact solution to the 3D Navier–Stokes equation [1], it was observed that the hydrodynamic variables like field velocities and pressure approach steady state. It would then be desirable to confirm this analytically. Starting from our 3D solutions, we specialize to 1D and 2D and show that analytically, there is indeed this ergodic behavior. In arriving at this result, we may have developed techniques that could be used to generalize the proof for 3D.

Our results will be derived from the following time evolution equation whose derivation is given in [5]:

$$
\begin{aligned}
f(r,p,t) =& f\left(r-\tfrac{pt}{m},p,0\right) + \int_0^t ds_1 e^{-s_1 L_o} \int dr' \left(\tfrac{\partial V(r-r')}{\partial r}\right) \tfrac{\partial}{\partial p} f_2^1(r,r',p,0) \\
&+ n_o \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} \int dr' \left(\tfrac{\partial V(r-r')}{\partial r}\right) \tfrac{\partial}{\partial p} \left(\tfrac{p}{m}\tfrac{\partial}{\partial r}\right) f_2^1(r,r',p,0) \\
&- n_o \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} \int dr' \left(\tfrac{\partial V(r-r')}{\partial r}\right) \tfrac{\partial}{\partial p} \left(\tfrac{p}{m}\tfrac{\partial}{\partial r}\right) \int dp' f_2^2(r,r',p,p',0) \\
&+ \tfrac{n_o^2}{2} \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} \int dr' \int dr'' \left(\tfrac{\partial V(r-r')}{\partial r}\right) \tfrac{\partial}{\partial p} \left(\tfrac{\partial V(r-r'')}{\partial r} \tfrac{\partial f_2^1(r,r',r'',p,0)}{\partial p}\right) \\
&+ \tfrac{n_o^2}{2} \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} \int dr' \left(\tfrac{\partial V(r-r')}{\partial r}\right) \tfrac{\partial}{\partial p} \left(\tfrac{\partial V(r-r')}{\partial r} \tfrac{\partial f_2^1(r,r',p,0)}{\partial p}\right) + S
\end{aligned}
\tag{1}
$$

where $S = O\left(\frac{\partial^n}{\partial p^n}\right)$ for $n = 3 \ldots \infty$.

We refer the reader to [2–5] for the definitions and conventions used in the above equation.

We drop S, but we are first obliged to show that normalization is preserved. To quickly show this we integrate over momentum

$$
\int dp f(r,p,t) = \int dp f\left(r - \frac{pt}{m}, p, 0\right) + \frac{n_o^2}{2} \int_0^t ds_1 \int_0^{s_1} ds_2 \int dp \left[\frac{s_2}{m} \frac{\partial}{\partial r} e^{-s_2 L_o} \int dr' \left(\frac{\partial V(r-r')}{\partial r}\right) \left(\frac{\partial V(r-r/)}{\partial r} \frac{\partial f_2^1(r,r',p,0)}{\partial p}\right)\right] \tag{2}
$$

Integrating over r

$$
\int dr \int dp f(r,p,t) = \int dr \int dp f\left(r - \frac{pt}{m}, p, 0\right) + \frac{n_o^2}{2} \int_0^t ds_1 \int_0^{s_1} ds_2 \int dp \int dr \left[\frac{s_2}{m} \frac{\partial}{\partial r} e^{-s_2 L_o} \int dr' \left(\frac{\partial V(r-r')}{\partial r}\right) \left(\frac{\partial V(r-r/)}{\partial r} \frac{\partial f_2^1(r,r',p,0)}{\partial p}\right)\right] \tag{3}
$$

The last integral of Eq. (3) is zero and

$$
\int dr \int dp f(r,p,t) = \int dr \int dp f\left(r - \frac{pt}{m}, p, 0\right) = 1 \tag{4}
$$

Normalization is indeed preserved. We may continue using our finite number of terms in Eq (1).

Suppose

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http://dx.doi.org/10.1016/j.rinp.2016.06.006
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$$f_2^1(r,r',p,0) = f(r)f(r')\varphi$$
$$f_2^2(r,r',p,p',0) = f(r)f(r')\varphi(p)\varphi(p') \tag{3}$$
$$f(r) = f(r') = 1$$

then we may write for uniform initial data

$$\begin{aligned} &f(x,y,x,p_x,p_y,p_z,t)\\ &=f\left(x-\frac{p_xt}{m},x-\frac{p_yt}{m},z-\frac{p_zt}{m},p_x,p_y,p_z,0\right)\\ &+\frac{n_o^2}{2}\int_0^t ds_1\int_0^{s_1} ds_2 e^{-s_2L_o}\left\{\int dr'\left(\frac{\partial V(r-r')}{\partial x}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_x^2}\right.\\ &+\int dr'\left(\frac{\partial V(r-r')}{\partial y}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_y^2}\\ &\left.+\int dr'\left(\frac{\partial V(r-r')}{\partial z}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_y^2}\right\} \end{aligned} \tag{4}$$

Differentiating Eq. (4) with respect to time,

$$\begin{aligned} \frac{\partial f(x,y,x,p_x,p_y,p_z,t)}{\partial t} &= +\frac{n_o^2}{2}\int_0^t ds_1 e^{-s_1L_o}\left\{\int dr'\left(\frac{\partial V(r-r')}{\partial x}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_x^2}\right.\\ &+\int dr'\left(\frac{\partial V(r-r')}{\partial y}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_y^2}\\ &\left.+\int dr'(\frac{\partial V(r-r')}{\partial z})^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_y^2}\right\} \end{aligned} \tag{5}$$

Now consider the formal operator equation

$$\int_0^t dse^{-sL_o} = (1-e^{-tL_o})\frac{1}{L_o} \tag{6}$$

so that

$$\begin{aligned} \frac{\partial f(x,y,x,p_x,p_y,p_z,t)}{\partial t} &= +\frac{n_o^2}{2}(1-e^{-tL_o})\frac{1}{L_o}\left\{\int dr'\left(\frac{\partial V(r-r')}{\partial x}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_x^2}\right.\\ &+\int dr'\left(\frac{\partial V(r-r')}{\partial y}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_y^2}\\ &\left.+\int dr'\left(\frac{\partial V(r-r')}{\partial z}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_y^2}\right\} \end{aligned} \tag{7}$$

When $t \to \infty$

$$\begin{aligned} \frac{\partial f(x,y,x,p_x,p_y,p_z,t)}{\partial t} &= +\frac{n_o^2}{2}\frac{1}{L_o}\left\{\int dr'\left(\frac{\partial V(r-r')}{\partial x}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_x^2}\right.\\ &+\int dr'\left(\frac{\partial V(r-r')}{\partial y}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_y^2}\\ &\left.+\int dr'(\frac{\partial V(r-r')}{\partial z})^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z}}{\partial p_y^2)}\right\} \end{aligned} \tag{8}$$

Next we use the Feynman definition of the inverse operator

$$\frac{1}{L_o} = \int_0^\infty dse^{-sL_o} \tag{9}$$

so that

$$\begin{aligned} \frac{\partial f(x,y,x,p_x,p_y,p_z,t)}{\partial t} &= +\frac{n_o^2}{2}\int_0^\infty dse^{-sL_o}\left\{\int dr'\left(\frac{\partial V(r-r')}{\partial x}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_x^2}\right.\\ &+\int dr'\left(\frac{\partial V(r-r')}{\partial y}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_y^2}\\ &\left.+\int dr'\left(\frac{\partial V(r-r')}{\partial z}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x},\boldsymbol{p_y},\boldsymbol{p_z})}{\partial p_y^2}\right\} \end{aligned} \tag{10}$$

Eq. (10) is still in 3D, and the three spatial integrals are equal. We will use the pair-potential

$$V(r-r') = gexp(-a((x-x')^2 + (y-y')^2 + (z-z')^2)) \tag{11}$$

Approach to a steady state in one and two dimensions

Using Eqs. (6) and (9)

$$\frac{\partial}{\partial t}f(x,p_x,t) = [(1-\exp(-tLo))]\int_o^\infty d\alpha e^{-\alpha Lo}\left\{\int dr'\left(\frac{\partial V(r-r')}{\partial x}\right)^2\frac{\partial^2\boldsymbol{\varphi}(\boldsymbol{p_x})}{\partial p_x^2}\right\} \tag{14}$$

All operators above act on right hand side expressions.
We have to evaluate our Feynman integral

$$F := \int_o^\infty d\alpha e^{-\alpha Lo}\left\{\int dx'\left(\frac{\partial V(x-x')}{\partial x}\right)^2\right\} \tag{15}$$

still using the 3D pair-potential.

We evaluate the space integral in a box of dimensions x from $-$W to W, y from $-$W to W, and z from $-$Z to Z, then put $y = x = 0$ to get

$$\begin{aligned} F &= \frac{(-g^2\pi erf(\sqrt{2a}W))^2}{8\sqrt{a}}\left[-\pi erf(\sqrt{2a}Z+\sqrt{2a}z)\sqrt{2}e^{2a(Z+z)^2}\right.\\ &- \sqrt{\pi}erf(\sqrt{2a}Z-\sqrt{2a}z)\sqrt{2}e^{2a(Z+z)^2} + 4Z\sqrt{a}e^{8azZ} - 4z\\ &\times\left.\sqrt{a}e^{8azZ}+4\sqrt{a}Z+4\sqrt{a}z\right]e^{-2a(Z+z)^2} \end{aligned} \tag{16}$$

which approaches zero as $W \to 0$. In one dimension, there is a rigorous approach to steady state.

The above procedure may be repeated for a box of dimension 2X by 2Y by 2Z, then evaluating the limit as $Z \to 0$, resulting in zero F. In 2D, there also exists an approach to steady state.

We conclude that in 1D and 2D, from Eq (14), $\frac{\partial f(x,p_x,t)}{\partial t} = 0$ as $t \to \infty$, there is an approach to a steady state for any initial momentum distribution. The final momentum distribution has a memory of the initial data.

For 3D, new techniques will be needed to show an approach to steady state, an open problem to be solved in a succeeding paper.

The rest of this paper will illustrate the consequences of the approach to steady state in 1D.

Hydrodynamic variables

We review our most general result from [4] for the time evolution of hydrodynamic variables: