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SWAGE algorithm for the cubic spline solution of nonlinear viscous Burgers' equation on a geometric mesh

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ABSTRACT

In this paper, we discuss the single sweep alternating group explicit (SWAGE) and Newton-SWAGE iteration methods to solve the non-linear ordinary differential equation $y'' = f(x, y, y')$ subject to given natural boundary conditions, along with a third order cubic spline numerical method on a geometric mesh. It is applicable to both singular and non-singular problems. The convergence of the SWAGE iteration method is discussed in detail. We compared the results of proposed SWAGE iteration method with the results of corresponding two parameter alternating group explicit (TAGE) iteration methods to demonstrate computationally the efficiency of the proposed method.

1. Introduction

The Burgers' equation was named after J.M. Burgers. The viscid Burgers' equation is a very important fluid dynamic problem. It is a nonlinear second order parabolic partial differential equation. Burgers' equation is a simplified form of the one-space dimensional Navier–Stokes equation. It possesses a fundamental quadratic non-linearity and is considered as an appropriate model for studying turbulence. It occurs in various areas of applied mathematics such as boundary layer formation, modeling of gas dynamics, traffic flow and shock waves, etc. The study of this equation has been considered important both for the conceptual understanding and for testing various numerical methods.

Many closed form analytical solutions have been obtained for Burgers' equation for various initial and boundary conditions by many researchers in the past. But these exact solutions are not effective for small values of viscosity. So, researchers are working on developing the numerical methods to efficiently and accurately solve the Burgers' equation with small viscosity. Various numerical methods based on finite difference approximations, finite element and spectral methods have been developed for the solution of the Burgers' equation.

In this paper, we discuss a new single sweep alternating group explicit (SWAGE) algorithm based on cubic spline approximation on a variable mesh for the numerical solution of nonlinear viscous Burgers' equation.

In the recent past, many authors (see [2–6,10–15]) have suggested various numerical methods based on cubic spline approximations for the solution of linear singular two point boundary value problems. In 1969 Fyfe [7] discussed the use of cubic spline for the solution of two point boundary value problems. Later, Albasiny and Hoskins [5,6] and Jain and Aziz [9] discussed both second- and fourth-order cubic spline methods using uniform mesh for the numerical solution of the non-linear two-point boundary value problems. In 1988, Chawla et al [8] developed fourth order cubic spline methods for linear singular boundary value problems. In 2003 Evans and Mohanty [18] developed a fourth-order accurate cubic spline alternating group explicit method for non-linear singular two-point boundary value problems. Recently Mohanty et al [22] have developed a third-order non-uniform mesh cubic spline method for the solution of non-linear singular two-point boundary value problems. On applying these higher order methods to linear and non-linear differential equations, we obtain large system of linear and non-linear equations respectively. In 1985, Evans [16,21] developed group explicit methods for solving large linear systems, which are suitable for use on parallel computers. Evans and Mohanty [18] have discussed Alternating group explicit method to solve nonlinear singular two point boundary value problems. Evans and Sukon [17], Mohanty et al [19] have further dis-

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cussed the two parameter AGE method. Evans and Mohanty [20] have developed a Coupled Alternating group explicit method for solving non-linear singular two point boundary value problems. Recently, Mohanty et al [23] have discussed the coupled reduced alternating group explicit (CRAGE) algorithm and sixth order off-step discretization for the solution of two point nonlinear boundary value problems.

In Section 2, we give the description of the cubic spline method and discuss its applications to singular linear and non-linear two point boundary value problems. In Section 3, we discuss the single sweep alternating group explicit (SWAGE) and Newton-SWAGE iterative method for solving the difference equations obtained on applying the cubic spline method to linear and nonlinear problems respectively. Further, we discuss the convergence of the SWAGE method in detail. In Section 4, we compare the performance of the proposed SWAGE and Newton-SWAGE iterative methods with the corresponding TAGE and Newton-TAGE iterative methods. Concluding remarks are given in Section 5.

2. Cubic spline approximation and application

Consider the general non-linear ordinary differential equation

$$y'' = f(x, y, y'), \quad a < x < b \tag{1}$$

subject to essential boundary conditions

$$y(a) = A, \quad y(b) = B, \tag{2}$$

where $-\infty < a < b < \infty$, A, B are finite constants.

We assume that for $x \in [a,b]$, $-\infty < y, z < \infty$

(i) $f(x,y,z)$ is continuous,
(ii) $\partial f/\partial y$ and $\partial f/\partial z$ exist and are continuous,
(iii) $\partial f/\partial y > 0$ and $|\partial f/\partial z| < W$ for some positive constant W.

These conditions ensure that the boundary value problem (1) and (2) possesses a unique solution (see Keller [1]).

To obtain a cubic spline solution of the boundary value problem (1) and (2), we discretize the interval $[0,1]$. Consider the solution interval $[0,1]$ with a non-uniform mesh such that $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$. Let $h_k = x_k - x_{k-1}$, $k = 1(1)N+1$ be the mesh size and $\sigma_k = h_{k+1}/h_k > 0$, $k = 1(1)N$ be the mesh ratio. Grid points are given by $x_i = x_0 + \sum_{k=1}^{i} h_k, i = 1(1)N+1$. Let $Y_k = y(x_k)$ be the exact solution of y at the grid point x_k and is approximated by y_k.

At each internal mesh point x_k, we denote:

$$M_k = y''(x_k) = f(x_k, y(x_k), y'(x_k)), \quad k = 0(1)N+1.$$

Given the values $y_0, y_1, \ldots, y_{N+1}$ of the function $y(x)$ at the mesh points $x_0, x_1, \ldots, x_{N+1}$ and the values of the second derivatives of y at the end points y''_0 and y''_{N+1}, there exists a unique interpolating cubic spline function $S(x)$ with the following properties:

(i) $S(x)$ coincides with a polynomial of degree three on each $[x_{k-1}, x_k]$, $k = 1(1)N+1$
(ii) $S(x) \in C^2[0,1]$ and
(iii) $S(x_k) = y_k$, $k = 0(1)N+1$

The interpolating cubic spline polynomial may be written as:

$$S(x) = \frac{(x_k - x)^3}{6h_k} M_{k-1} + \frac{(x - x_{k-1})^3}{6h_k} M_k + \left(y_{k-1} - \frac{h_k^2}{6} M_{k-1}\right)\frac{(x_k - x)}{h_k} + \left(y_k - \frac{h_k^2}{6} M_k\right)\frac{(x - x_{k-1})}{h_k}, \quad x_{k-1} \leqslant x \leqslant x_k, \ k = 1(1)N+1 \tag{3}$$

At each grid point x_k, we denote

$$P_k = \sigma_k^2 + \sigma_k - 1,$$
$$Q_k = (\sigma_k + 1)(\sigma_k^2 + 3\sigma_k + 1),$$
$$R_k = \sigma_k(1 + \sigma_k - \sigma_k^2),$$
$$S_k = \sigma_k(1 + \sigma_k)$$

Let, $G_k = \frac{\partial f}{\partial Y'_k}$ etc.

At each grid point x_k, the differential equation (1) may be written as

$$Y''_k = f(x_k, Y_k, Y'_k) \equiv f_k$$

Using Taylor series expansion, we first obtain

$$Y_{k+1} - (1 + \sigma_k)Y_k + \sigma_k Y_{k-1} = \frac{h_k^2}{12}[P_k f_{k+1} + Q_k f_k + R_k f_{k-1}] + T_k, \quad k = 1(1)N$$

where $T_k = O\left(h_k^5\right)$.

We consider the following approximations:

Let,

$$\bar{m}_k = \overline{Y}'_k = \left(Y_{k+1} - (1 - \sigma_k^2)Y_k - \sigma_k^2 Y_{k-1}\right)/(h_k\sigma_k(\sigma_k + 1)), \tag{5.1}$$

$$\bar{m}_{k+1} = \overline{Y}'_{k+1} = \frac{(1 + 2\sigma_k)Y_{k+1} - (1 + \sigma_k)^2 Y_k + \sigma_k^2 Y_{k-1}}{h_k S_k} \tag{5.2}$$

$$\bar{m}_{k-1} = \overline{Y}'_{k-1} = \frac{-Y_{k+1} + (1 + \sigma_k)^2 Y_k - \sigma_k(2 + \sigma_k)Y_{k-1}}{h_k S_k} \tag{5.3}$$

$$\bar{f}_k = f(x_k, Y_k, \bar{m}_k), \tag{5.4}$$

$$\bar{f}_{k\pm1} = f(x_{k\pm1}, Y_{k\pm1}, \bar{m}_{k\pm1}), \tag{5.5}$$

$$\hat{m}_k = \widehat{Y}'_k = \bar{m}_k - \frac{\sigma_k h_k}{6(1 + \sigma_k)}(\bar{f}_{k+1} - \bar{f}_{k-1}), \tag{5.6}$$

$$\widehat{Y}'_{k+1} = \frac{Y_{k+1} - Y_k}{\sigma_k h_k} + \frac{\sigma_k h_k}{6}(\bar{f}_k + 2\bar{f}_{k+1}), \tag{5.7}$$

$$\widehat{Y}'_{k+1} = \frac{Y_k - Y_{k-1}}{h_k} - \frac{h_k}{6}(\bar{f}_k + 2\bar{f}_{k-1}), \tag{5.8}$$

$$\hat{f}_{k\pm1} = f\left(x_{k\pm1}, Y_{k\pm1}, \widehat{Y}'_{k\pm1}\right), \tag{5.9}$$

$$\hat{f}_k = f\left(x_k, Y_k, \widehat{Y}'_k\right), \tag{5.10}$$

Then the cubic spline method with order of accuracy three for the differential Eq. (1) may be written as:

$$Y_{k+1} - (1 + \sigma_k)Y_k + \sigma_k Y_{k-1} = \frac{h_k^2}{12}[P_k \hat{f}_{k+1} + Q_k \hat{f}_k + R_k \hat{f}_{k-1}] + \widehat{T}_k, \quad k = 1(1)N \tag{6}$$

where $\widehat{T}_k = O\left(h_k^5\right)$ (See Mohanty et al [22]) with $y_0 = Y_0 = A$ and $y_{N+1} = Y_{N+1} = B$. If the differential equation (1) is linear we apply the SWAGE iterative method to obtain the solution, and if it is non-linear, we use the Newton-SWAGE iterative method.

Now, we consider the application of the cubic spline method (6) to the linear singular equation

$$y'' = D(x)y' + E(x)y + f(x), \quad 0 < x < 1 \tag{7}$$

and non-linear singular equation

$$\upsilon y'' = B(x)y' + yy' + C(x)y + g(x), \quad 0 < x < 1 \tag{8}$$

where $\upsilon = R_e^{-1} > 0$ is a constant and $D(x) = -\alpha/x$ and $E(x) = \alpha/x^2$, $B(x) = -\alpha\upsilon/x$ and $E(x) = \alpha\upsilon/x^2$.

For $\alpha = 1$ and 2, the linear singular Eq. (7) becomes cylindrical and spherical problems, respectively, and for $\alpha = 0$, 1 and 2, the non-linear singular problem (8) represents steady-state Burger's equation in Cartesian, cylindrical and spherical coordinates respectively.

Now applying the difference formula (6) to the singular Eq. (7) and neglecting the local truncation error, we obtain