Results in Physics 5 (2015) 111–114

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.journals.elsevier.com/results-in-physics

A new reliable analytical solution for strongly nonlinear oscillator with cubic and harmonic restoring force

Md. Alal Hosen [a,b,*], M.S.H. Chowdhury [a]

[a] Department of Science in Engineering, Faculty of Engineering, International Islamic University Malaysia, Jalan Gombak, 53100 Kuala Lumpur, Malaysia
[b] Department of Mathematics, Rajshahi University of Engineering and Technology, Rajshahi 6204, Bangladesh

ARTICLE INFO

Article history:
Received 23 March 2015
Accepted 24 April 2015
Available online 2 May 2015

Keywords:
Analytical technique
Strongly nonlinear oscillator
Approximate solutions
Harmonic balance method
Power series solution

ABSTRACT

In the present paper, a complicated strongly nonlinear oscillator with cubic and harmonic restoring force, has been analysed and solved completely by harmonic balance method (HBM). Investigating analytically such kinds of oscillator is very difficult task and cumbersome. In this study, the offered technique gives desired results and to avoid numerical complexity. An excellent agreement was found between approximate and numerical solutions, which prove that HBM is very efficient and produces high accuracy results. It is remarkably important that, second-order approximate results are almost same with exact solutions. The advantage of this method is its simple procedure and applicable for many other oscillatory problems arising in science and engineering.

Introduction

Nonlinear oscillations are important fact in physical science, mechanical structures and other engineering problems. Naturally, all differential equations involving engineering and physical phenomena are nonlinear. The methods of solutions of linear differential equations are comparatively easy and well established. On the contrary, the techniques of solutions of nonlinear differential equations (NDEs) are less available and have no exact solution and, in general, linear approximations are frequently used. Nowadays, NDEs have been the subject of all-embracing studies in various branches of nonlinear science and engineering. A special class of analytical solutions named strongly nonlinear oscillator with cubic and harmonic restoring force has a lot of importance, because, most of the phenomena that arise in mathematical physics and engineering fields can be described by NDEs. Therefore, investigating strongly nonlinear oscillator with cubic and harmonic restoring force solutions is becoming increasingly attractive in nonlinear sciences. Moreover, obtaining exact solutions for nonlinear oscillatory problems has many difficulties. It is very difficult to solve nonlinear problems and in general it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear problem. To overcoming the shortcomings, many new analytical techniques have been successfully developed by diverse groups of mathematicians and physicists, such as, Perturbation Method [1], Homotopy Perturbation Method [2], Modified Homotopy Perturbation Method [3,4], Rational Homotopy Perturbation Method [5], He's Homotopy Perturbation Method [6], Modified He's homotopy Perturbation Method [7], Asymptotic Method [8–11], Optimal Iteration Perturbation Method [12], Generalization of Modified Differential Transforms Method [13–16], and so on. Several other authors used many powerful analytical methods in the field of approximate solutions especially for strongly nonlinear oscillators like Max–Min Approach Method [17,18], Algebraic Method [19], Parameter Expansion Method and Variational Iteration Method [20–22], Amplitude Frequency Formulation Method [23], Energy Balance Method [24,25], He's Energy Balance Method [26,27], Rational Energy Balance Method [28], Rational Harmonic Balance Method [29], Residue Harmonic Balance Method [30–33], Newton-harmonic Balancing Approach [34], and so on for solving NDEs. The HBM is another technique for solving strongly nonlinear systems. Borges et al. [35] and Bobylev et al. [36] first provided overviews of HBM. Mickens [37–39] was first applied HBM in truly nonlinear oscillators. Due to his contribution he is known as father of HBM. Afterwards, Belendez et al. [40] and others researchers [41–43] has significantly improved the HBM. The HBM provides a general technique for calculating approximations to the periodic solutions of linear and NDEs. It corresponds to a truncated Fourier series and allows for the systematic determination of the coefficients to various harmonics and the angular frequency. The significance of the method is that it may be applied to differential equations for which the nonlinear terms are not small.

* Corresponding author.
E-mail addresses: alal_ruet@yahoo.com (Md. Alal Hosen), sazzadbd@iium.edu.my (M.S.H. Chowdhury).

http://dx.doi.org/10.1016/j.rinp.2015.04.004
2211-3797/

For strongly nonlinear oscillatory problems, perturbation techniques and other analytical methods mentioned above do not provide expected results. Moreover, a set of difficult nonlinear algebraic equations appear when HBM is formulated. But in classical HBM and some modifications of HBM discussed above there is no clear idea for solving these complicated nonlinear higher order algebraic equations especially in the case of a large oscillation. To overcome these aforementioned issues, we have offered an analytical technique based on modified harmonic balance method (MHBM) for solving strongly nonlinear systems. In this technique, a new parameter (small) has been introduced to solve the nonlinear algebraic equations by a power series solution when the nonlinear terms of the original equation are neither significant nor small. The higher order approximations (mainly second approximation) have been obtained for strongly nonlinear oscillators with cubic and harmonic restoring force. Comparison of the obtaining results with its exact solutions which show that the proposed method is effective and convenient for solving these analytical results. The advantage of the MHBM is that the solution gives more correct results than corresponding many existing solutions.

The objective of this article is to employ the MHBM to find new approximate solutions of strongly nonlinear oscillator with cubic and harmonic restoring force. The MHBM is very easy, direct, concise and simple to implement compared to other existing methods.

The rest of the article is organized as follows: In Section 2, we give the outline of the method. In Section 3, we implement this method to strongly nonlinear oscillator with cubic and harmonic restoring force. Finally, in Section 4, concluding remarks are given.

The method

Let us consider a nonlinear differential equation

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x) \quad \text{and the initial conditions } [x(0) = A_0,\ \dot{x}(0) = 0], \tag{1}$$

where $f(x)$ is a nonlinear function such that $f(-x) = -f(x)$, $\omega_0 \geqslant 0$ and ε is a constant.

Consider a periodic solution of Eq. (1) is in the form

$$x = A_0(\rho\cos(\omega t) + u\cos(3\omega t) + v\cos(5\omega t) + w\cos(7\omega t) + z \times \cos(9\omega t)\cdots), \tag{2}$$

where A_0, ρ and ω^2 are constants. If $\rho = 1 - u - v - \cdots$ and the initial phase $(\omega t)_0 = 0$, solution Eq. (2) readily satisfies the given initial condition Eq. (1).

Substituting Eq. (2) into Eq. (1) and expanding $f(x)$ in a Fourier series, it converts to an algebraic identity as follows:

$$A_0[\rho(\omega_0^2 - \omega^2)\cos(\omega t) + u(\omega_0^2 - 9\omega^2)\cos(3\omega t) + \cdots] = -\varepsilon[F_1(A_0, u, \cdots)\cos(\omega t) + F_3(A_0, u, \cdots)\cos(3\omega t) + \cdots] \tag{3}$$

By comparing the coefficients of equal harmonics of Eq. (3), the following nonlinear algebraic equations are found

$$\rho(\omega_0^2 - \omega^2) = -\varepsilon F_1, \quad u(\omega_0^2 - 9\omega^2) = -\varepsilon F_3, \quad v(\omega_0^2 - 25\omega^2) = -\varepsilon F_5, \cdots \tag{4}$$

With the help of the first equation, ω^2 is eliminated from the rest of Eq. (4). Thus Eq. (4) takes the following form

$$\rho\omega^2 = \rho\omega_0^2 + \varepsilon F_1, \quad 8\omega_0^2 u\rho = \varepsilon(\rho F_3 - 9uF_1), \quad 24\omega_0^2 v\rho = \varepsilon(\rho F_5 - 25vF_1), \cdots \tag{5}$$

By substitution $\rho = 1 - u - v - \cdots$, and simplification, second-, third- equations of Eq. (5) take the following form

$$u = G_1(\omega_0, \varepsilon, A_0, u, v, \cdots, \lambda_0), \quad v = G_2(\omega_0, \varepsilon, A_0, u, v, \cdots, \lambda_0), \cdots, \tag{6}$$

where $G_1, G_2, \cdots$ exclude respectively the linear terms of $u, v, \cdots$.

Whatever the values of ε, ω_0 and A_0 there exists a parameter $\lambda_0(\varepsilon, \omega_0, A_0) \ll 1$, such that $u, v, \cdots$ are expandable in following power series in terms of λ_0 as

$$u = U_1\lambda_0 + U_2\lambda_0^2 + \cdots, \quad v = V_1\lambda_0 + V_2\lambda_0^2 + \cdots, \quad \cdots \tag{7}$$

where $U_1, U_2, \cdots, V_1, V_2, \cdots$ are constants.

Finally, substituting the values of $u, v, \cdots$ from Eq. (7) into the first equation of Eq. (5), ω is determined. This completes the determination of all related functions for the proposed periodic solution as given in Eq. (2).

Example

In the present paper, we consider a strongly nonlinear oscillator with cubic and harmonic restoring force

$$\ddot{x} + x + ax^3 + b\sin(x) = 0, \tag{8}$$

where a and b are known constants and dot denotes derivative with respect to time t. The initial conditions are given by

$$[x(0) = A_0, \dot{x}(0) = 0]. \tag{9}$$

where substitution of approximation

$$\sin(x) = x - \frac{x^3}{6}. \tag{10}$$

into Eq. (8) yields

$$\ddot{x} + x + ax^3 + b\left(x - \frac{x^3}{6}\right) = 0. \tag{11}$$

From Eq. (2) the first-order approximation solution of Eq. (11) is

$$x = A_0\cos(\omega_1 t) \tag{12}$$

Now substituting Eq. (12) into Eq. (11) and setting the coefficient of $\cos(\omega_1 t)$ equal to zero the following algebraic equation is obtained

$$1 + \frac{3aA_0^2}{4} + b - \frac{A_0^2 b}{8} - \omega_1^2 = 0. \tag{13}$$

Thus from Eq. (13) the first-order approximate angular frequency is

$$\omega_1 = \sqrt{1 + \frac{3aA_0^2}{4} + b - \frac{A_0^2 b}{8}}. \tag{14}$$

Therefore the first-order approximation solution of Eq. (9) is Eq. (12) *i.e.* $x = A_0\cos(\omega_1 t)$ where ω_1 is given by Eq. (14).

Let us consider a second-order approximation solution

$$x = A_0\cos(\omega_2 t) + A_0 u(\cos(3\omega_2 t) - \cos(\omega_2 t)) \tag{15}$$

Substituting Eq. (15) into the Eq. (11) and then equating the coefficients of $\cos(\omega_2 t)$ and $\cos(3\omega_2 t)$, the following equations are

$$1 + 3aA_0^2/4 + b - A_0^2 b/8 - \omega_2^2 - u - 3aA_0^2 u/2 - bu + A_0^2 bu/4 + \omega_2^2 u + 9aA_0^2 u^2/4 - 3A_0^2 bu^2/8 - 3aA_0^2 u^3/2 + A_0^2 bu^3/4 = 0, \tag{16}$$

$$aA_0^2/4 - A_0^2 b/24 + u + 3aA_0^2 u/4 + bu - A_0^2 bu/8 - 9\omega_2^2 u - 9aA_0^2 u^2/4 + 3A_0^2 bu^2/8 + 2aA_0^2 u^3 - A_0^2 bu^3/3 = 0 \tag{17}$$