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A modified Numerov method for solving singularly perturbed differential–difference equations arising in science and engineering

P. Pramod Chakravarthy *, R. Nageshwar Rao

Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur 440010, India

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ABSTRACT

In this paper a modified fourth order Numerov method is presented for singularly perturbed differential–difference equation of mixed type, i.e., containing both terms having a negative shift and terms having positive shift. Similar boundary value problems are associated with expected first exit time problems of the membrane potential in the models for the neuron. To handle the negative and positive shift terms, we construct a special type of mesh, so that the terms containing shift lie on nodal points after discretization. The proposed finite difference method works nicely when the shift parameters are smaller or bigger to perturbation parameter. An extensive amount of computational work has been carried out to demonstrate the proposed method and to show the effect of shift parameters on the boundary layer behavior or oscillatory behavior of the solution of the problem.

Introduction

Any system involving feedback control will almost involve time delays. These arise because a finite time is required to sense the information and then react to it. A singularly perturbed differential–difference equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involves at least one delay term. Such problems arise frequently in the mathematical modeling of various physical and biological phenomena like optically bistable devices [1], description of the human pupil reflex [2], a variety of models for physiological processes or diseases and variational problems in control theory [3,4], the first exit time problem in the modeling of the activation of neuronal variability [5]. Lange and Miura [5,6] gave an asymptotic approach in the study of a class of boundary value problems for linear second order differential–difference equations in which the highest order derivative is multiplied by a small parameter. An extensive numerical work had been initiated by Kadalbajoo et al. [7–11]. In [12] Ramos has presented a variety of exponential methods for the numerical solution of linear ordinary differential–difference equations with a small delay based on piecewise analytical solutions of advection–reaction–diffusion operators. In [13], the authors Jugal Mohapatra, Srinivasan Natesan constructed a numerical method for a class of singularly perturbed differential–difference equations with small delay.

In this paper we modified the fourth order Numerov method and applied to singularly perturbed differential–difference equations of mixed type. To handle the negative and positive shift terms, we construct a special type of mesh, so that the terms containing shift lie on nodal points after discretization. The proposed finite difference method works nicely when the shift parameters are smaller or bigger to perturbation parameter. An extensive amount of computational work has been carried out to demonstrate the proposed method and to show the effect of shift parameters on the boundary layer behavior and oscillatory behavior of the solution of the problem.

Modified fourth order Numerov method

We consider a linear singularly perturbed differential–difference equation of mixed type i.e., equation containing both the negative and positive shift terms.

$$\varepsilon^2 y''(x) + \alpha(x)y(x-\delta) + \omega(x)y(x) + \beta(x)y(x+\eta) = f(x) \tag{1}$$

on $0 < x < 1$, $0 < \varepsilon << 1$, subject to the interval and boundary conditions

$$y(x) = \phi(x) \quad \text{for} \quad -\delta \leqslant x \leqslant 0 \quad \text{and} \quad y(x) = \psi(x) \quad \text{for} \quad 1 \leqslant x \leqslant 1+\eta \tag{2}$$

where $\alpha(x)$, $\omega(x)$, $\beta(x)$, $f(x)$, $\phi(x)$ and $\psi(x)$ are smooth functions, δ and η are the small shifting parameters. For a function $y(x)$ to constitute a smooth solution to the problem (1), (2) it must be continuous in the interval [0,1] and be continuously differentiable in the interval (0,1). For the shifts δ, η equal to zero and if

* Corresponding author. Tel.: +91 712 2801404; fax: +91 712 2223969/2223230.
E-mail addresses: ppchakravarthy@mth.vnit.ac.in (P. Pramod Chakravarthy), nrao_ragi@yahoo.co.in (R. Nageshwar Rao).

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$\alpha(x) + \omega(x) + \beta(x) < 0$ on the interval [0,1], then the solution exhibits boundary layers at both the ends of the interval [0,1].

We rearrange the differential Eq. (1) and (2) as

$$\varepsilon^2 y''(x) = g(x, y(x), y(x-\delta), y(x+\eta)) \tag{3}$$

where $g(x, y(x), y(x-\delta), y(x+\eta)) = f(x) - \alpha(x)y(x-\delta) - \omega(x)y(x) - \beta(x)y(x+\eta)$

Now, we construct a special type of mesh so that the terms containing the shift parameters lie on the nodal points after discretization. We divide the interval [0, 1] into N equal parts by choosing the mesh parameter h such that $h = \frac{\delta}{k} = \frac{\eta}{\ell}$, where k and ℓ are positive integers chosen such that $1 \leqslant k, \ell \leqslant N$.

At $x = x_i$, the above differential equation can be written as

$$\varepsilon^2 y''(x_i) = g(x_i, y(x_i), y(x_i-\delta), y(x_i+\eta)) = g_i, \tag{4}$$

where $g_i = f_i - \alpha_i y_{i-k} - \omega_i y_i - \beta_i y_{i+\ell}$,

$y_i = y(x_i), f_i = f(x_i), \alpha_i = \alpha(x_i), \omega_i = \omega(x_i), \beta_i = \beta(x_i)$.

Now, we consider the fourth order Numerov finite difference method [14] to solve the Eq. (4) and this equation is approximated by the following finite difference scheme:

$$\frac{\varepsilon^2}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = \frac{1}{12}(g_{i-1} + 10g_i + g_{i+1}) \tag{5}$$

The boundary conditions can be written as

$$y_i = \phi_i; -k \leqslant i \leqslant 0 \quad \text{and} \quad y_i = \psi_i; N \leqslant i \leqslant N + \ell \tag{6}$$

where $\phi_i = \phi(x_i)$ and $\psi_i = \psi(x_i)$.

Using the definition of g_i in Eq. (5), we get the following fourth order finite difference scheme.

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^* y_{i-k-1} + F_i^* y_{i-k} + G_i^* y_{i-k+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} + G_i^{**} y_{i+\ell+1} = R_i \tag{7}$$

where

$$E_i = \frac{12\varepsilon^2}{h^2} + \omega_{i-1}, F_i = -\frac{24\varepsilon^2}{h^2} + 10\omega_i, G_i = \frac{12\varepsilon^2}{h^2} + \omega_{i+1},$$
$$E_i^* = \alpha_{i-1}, F_i^* = 10\alpha_i, G_i^* = \alpha_{i+1}, E_i^{**} = \beta_{i-1}, F_i^{**} = 10\beta_i,$$
$$G_i^{**} = \beta_{i+1}, R_i = f_{i-1} + 10f_i + f_{i+1}$$

Using (6), the difference scheme (7) can be written as

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} + G_i^{**} y_{i+\ell+1} = R_i - E_i^* \phi_{i-k-1} - F_i^* \phi_{i-k} - G_i^* \phi_{i-k+1} \quad \text{for} \quad 1 \leqslant i \leqslant k-1$$

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} + G_i^{**} y_{i+\ell+1} + G_i^* y_{i-k+1} = R_i - E_i^* \phi_{i-k-1} - F_i^* \phi_{i-k} \quad \text{for} \quad i = k$$

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} + G_i^{**} y_{i+\ell+1} + G_i^* y_{i-k+1} + F_i^* y_{i-k} = R_i - E_i^* \phi_{i-k-1} \quad \text{for} \quad i = k+1$$

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} + G_i^{**} y_{i+\ell+1} + G_i^* y_{i-k+1} + F_i^* y_{i-k} + E_i^* y_{i-k-1} = R_i \quad \text{for} \quad k+2 \leqslant i \leqslant N-\ell-2$$

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} + G_i^* y_{i-k+1} + F_i^* y_{i-k} + E_i^* y_{i-k-1} = R_i - G_i^{**} \psi_{i+\ell+1} \quad \text{for} \quad i = N-\ell-1$$

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + G_i^* y_{i-k+1} + F_i^* y_{i-k} + E_i^* y_{i-k-1} = R_i - G_i^{**} \psi_{i+\ell+1} - F_i^{**} \psi_{i+l} \quad \text{for} \quad i = N-\ell$$

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + G_i^* y_{i-k+1} + F_i^* y_{i-k} + E_i^* y_{i-k-1} = R_i - G_i^{**} \psi_{i+\ell+1} - F_i^{**} \psi_{i+\ell} - E_i^{**} \psi_{i+\ell-1} \quad \text{for} \quad N-\ell+1 \leqslant i \leqslant N-1$$

The above system of equations along with the boundary conditions $y_0 = \phi_0$ and $y_N = \psi_N$ is solved for y_i, i = 0, 1, 2,....., N by Gauss elimination method with partial pivoting. In fact, any numerical method or analytical method can be used to solve the above system of equations for y_i.

Table 1
Numerical solution of example 1 for $\delta = 0.03$, $\eta = 0.07$.

N→	100	200	300	400	500
ε↓					
2^{-1}	6.1000e−007	1.6000e−007	7.0000e−008	4.0000e−008	3.0000e−008
2^{-2}	6.7400e−006	1.6900e−006	7.5000e−007	4.2000e−007	2.7000e−007
2^{-3}	5.0780e−005	1.2710e−005	5.6500e−006	3.1800e−006	2.0300e−006
2^{-4}	2.9686e−004	7.4640e−005	3.3200e−005	1.8690e−005	1.1960e−005
2^{-5}	1.6272e−003	4.1624e−004	1.8578e−004	1.0466e−004	6.7020e−005
2^{-6}	6.6542e−003	1.8089e−003	8.1653e−004	4.6180e−004	2.9629e−004

Table 2
Numerical solution of example 2 for $\delta = 0.07$, $\eta = 0.03$.

N→	100	200	300	400	500
ε↓					
2^{-1}	1.4510e−005	3.6300e−006	1.6100e−006	9.0000e−007	5.8000e−007
2^{-2}	9.3650e−005	2.3420e−005	1.0410e−005	5.8600e−006	3.7500e−006
2^{-3}	5.2693e−004	1.3196e−004	5.8660e−005	3.3000e−005	2.1130e−005
2^{-4}	2.5668e−003	6.4570e−004	2.8731e−004	1.6168e−004	1.0349e−004
2^{-5}	9.9696e−003	2.5626e−003	1.1448e−003	6.4511e−004	4.1321e−004
2^{-6}	3.1132e−002	8.7328e−003	3.9629e−003	2.2454e−003	1.4419e−003

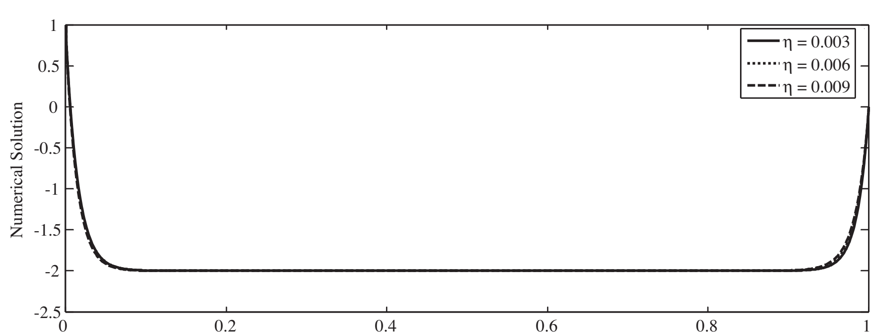

Fig. 1. The numerical solution of example 1 with $\varepsilon = 0.01$ and $\delta = 0.005$ for different values of η.