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Letter

New exact solution for Rayleigh–Stokes problem of Maxwell fluid in a porous medium and rotating frame

1. Introduction

Stokes' first problem for the flat plate originated in 1851, and is also known as the Rayleigh–Stokes problem. This problem for non-Newtonian fluid has received much attention due to its practical applications in industry, geophysics, chemical and petroleum engineering [1]. Some investigations are notably important in industries related to paper, food stuff, personal care products, textile coating and suspension solutions.

For the Rayleigh–Stokes problem, we shall consider an infinitely long flat plate above which a fluid exists. Initially, both the fluid and plate are at rest and suddenly, the plate is jerked into its plane with a constant velocity. For Newtonian fluids and by using a simple transformation, an elegant solution was obtained for this problem by Zierep [2] and Soundalegkar [3]. More recently, Tan and Masuoka [4,5], Fetecau and Corina Fetecau [6], Zierep and Fetecau [7,8], Hayat et al. [9–11], Fetecau and Corina Fetecau [12] and Fetecau et al. [13] have studied the problem for different types of non-Newtonian fluids.

The non-Newtonian fluids have been mainly classified under the differential, rate and integral types. The Maxwell fluids are the subclass of non-Newtonian fluids and are the simplest subclass of rate type fluids which take the relaxation phenomena into consideration. It was employed to study various problems due to their relatively simple structure. Moreover, one can reasonably hope to obtain exact solutions from this type of Maxwell fluid. This motivates us to choose the Maxwell model in this study. The exact solutions are important as these provide standard for checking the accuracies of many approximate solutions which can be numerical or empirical. They can also be used as tests for verifying numerical schemes that are developed for studying more complex flow problems.

Exact solution of the problem is given by using the Fourier sine and Laplace transforms method. This method has already been successfully applied by various workers, for example, Fetecau et al. [13,14] and Christov and Jordan [15]. Justifiably, the traditional Fourier sine and Laplace transforms method has the following important features. It is a very powerful technique for solving these kinds of problems, which literally transforms the original linear differential equation into an elementary algebraic expression. More importantly, the transformation avoids the omission of a critical term from the resulting subsidiary equation.

The objective of the present work is to establish a new exact solution for a magnetohydrodynamic (MHD) Maxwell fluid in a porous medium and rotating frame. Here we examine the rotating and MHD flow over a suddenly moved flat plate. Constitutive equations of a Maxwell fluid are used. Modified Darcy's law has been utilized. The solution to the resulting problem is generated by Fourier sine and Laplace transforms technique. The graphs of the velocity profiles are plotted in order to illustrate the variations of embedded flow parameters with respect to the velocity profiles. Interestingly, the results of many existing situations (see [12,16,17]) are shown as the special cases of the present study.

2. Formulation of the problem

We choose a Cartesian coordinate system by considering an infinite plate at $z = 0$. An incompressible fluid which occupies the porous space is conducting electrically by the exertion of an applied magnetic field B_0, which is parallel to the $z-$ axis. The electric field is not taken into consideration and the magnetic Reynolds number is small and such that the induced magnetic field is not accounted for. The Lorentz force $J \times B_0$ under these conditions is equal to $-\sigma B_0^2 V$. Here J is the current density, V is the velocity field, σ is the electrical conductivity of fluid. Both plate and fluid possess solid body rotation with a uniform angular velocity Ω about the z-axis.

The governing equations are

$$div V = 0, \tag{1}$$

$$\rho\left[\frac{\partial V}{\partial t} + (V\cdot\nabla)V + 2\Omega\times V + \Omega\times(\Omega\times r)\right] = -\nabla p + div S - \sigma B_0^2 V + R, \tag{2}$$

where ρ is the fluid density, r is a radial vector with $r^2 = x^2 + y^2$, p is the pressure, S is the extra stress tensor and R is Darcy's resistance.

The constitutive relationships for Maxwell fluid are

$$T = -pI + S,$$
$$S + \lambda\left[\frac{dS}{dt} - LS - SL^T\right] = \mu A, \tag{3}$$

where T is the Cauchy stress tensor, I is the identity tensor, L is the velocity gradient, $A = L + L^T$ is the first Rivlin–Eriksen tensor, λ the relaxation and μ is the dynamic viscosity of fluid and $\frac{d}{dt}$ indicates the material derivative.

According to Tan and Masuka [4], Darcy's resistance in an Oldroyd-B fluid satisfying the following expression:

$$\left(1+\lambda\frac{\partial}{\partial t}\right)R = -\frac{\mu\phi}{k}\left(1+\lambda_r\frac{\partial}{\partial t}\right)V, \tag{4}$$

where λ_r is the retardation time, ϕ is the porosity and k is the permeability of the porous medium. For Maxwell fluid $\lambda_r = 0$ and hence

$$\left(1+\lambda\frac{\partial}{\partial t}\right)R = -\frac{\mu\phi}{k}V. \tag{5}$$

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We seek a velocity field of the form

$$V = (u(z,t), \upsilon(z,t), w(z,t)), \tag{6}$$

which together with Eq. (1) yield $w = 0$. By using the Eqs. (2), (3), and (6) we arrive at

$$\rho\left(\frac{\partial u}{\partial t} - 2\Omega\upsilon\right) = -\frac{\partial \hat{p}}{\partial x} + \frac{\partial S_{xz}}{\partial z} - \sigma B_0^2 u + R_x, \tag{7}$$

$$\rho\left(\frac{\partial \upsilon}{\partial t} + 2\Omega u\right) = -\frac{\partial \hat{p}}{\partial y} + \frac{\partial S_{xz}}{\partial z} - \sigma B_0^2 \upsilon + R_y, \tag{8}$$

where

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)S_{xz} = \mu\frac{\partial u}{\partial z}, \tag{9}$$

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)S_{yz} = \mu\frac{\partial \upsilon}{\partial z}. \tag{10}$$

The R_x and R_y are x- and y-components of Darcy's resistance R, and z-component of Eq. (2) indicates that $\hat{p} \neq \hat{p}(z)$ and the modified pressure $\hat{p}$ is $\hat{p} = p - \frac{\rho}{2}\Omega^2 r^2$.

Invoking Eqs. (5), (9), and (10) in Eqs. (7) and (8) and then neglecting the pressure gradient we now obtain the coupled governing equations as

$$\rho\left(1 + \lambda\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial t} - 2\Omega\upsilon\right) + \sigma B_0^2\left(1 + \lambda\frac{\partial}{\partial t}\right)u = \mu\frac{\partial^2 u}{\partial z^2} - \frac{\mu\phi}{k}u, \tag{11}$$

$$\rho\left(1 + \lambda\frac{\partial}{\partial t}\right)\left(\frac{\partial \upsilon}{\partial t} + 2\Omega u\right) + \sigma B_0^2\left(1 + \lambda\frac{\partial}{\partial t}\right)\upsilon = \mu\frac{\partial^2 \upsilon}{\partial z^2} - \frac{\mu\phi}{k}\upsilon, \tag{12}$$

where the appropriate initial and boundary conditions are

$$u = \upsilon = 0 \quad \text{at} \quad t = 0, z > 0 \tag{13}$$

$$u(0,t) = U, \ \upsilon(0,t) = 0 \quad \text{for} \quad t > 0, \tag{14}$$

$$u, \frac{\partial u}{\partial z}, \ \upsilon, \frac{\partial \upsilon}{\partial z} \to 0 \quad \text{as} \quad z \to \infty; \quad t > 0. \tag{15}$$

3. Solution of the problem

Letting $F = u + i\upsilon$ in Eqs. (11) and (12), the problem is reduced by combining these two equations as

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\frac{\partial F}{\partial t} + \left(2i\Omega + \frac{\sigma B_0^2}{\rho}\right)\left(1 + \lambda\frac{\partial}{\partial t}\right)F + \frac{\nu\phi}{k}F = \nu\frac{\partial^2 F}{\partial z^2}, \tag{16}$$

where ν is the kinematic viscosity. The appropriate boundary and initial conditions are

$$F(0,t) = U, \quad t > 0, \tag{17}$$

$$F(z,0) = \frac{\partial F(z,0)}{\partial t} = 0, \quad z > 0,$$

$$F(z,t) = \frac{\partial F(z,t)}{\partial z} \to 0 \quad \text{as} \quad z \to \infty, \quad t > 0.$$

In order to solve the linear partial differential equation (16) with initial and boundary conditions (17), we shall use the Fourier sine and Laplace transforms technique [13–15]. For a greater generality, we consider the boundary condition $F(0,t) = U(t)$ with $U(0) = 0$ and apply the Fourier sine transform with respect to z.

We thus obtain the result as follows:

$$\lambda\frac{\partial^2 F_s(\eta,t)}{\partial t^2} + [1 + \lambda c]\frac{\partial F_s(\eta,t)}{\partial t} + \left[\nu\left(\eta^2 + \frac{\phi}{k}\right) + c\right]F_s(\eta,t)$$

$$= \eta\sqrt{\frac{2}{\pi}}\nu U(t); \quad t > 0, \tag{18}$$

where $c = 2i\Omega + \frac{\sigma B_0^2}{\rho}$ and the Fourier sine transform $F_s(\eta,t)$ of $F(z,t)$ has to satisfy the following conditions:

$$F_s(\eta,0) = \frac{\partial F_s(\eta,0)}{\partial t} = 0; \quad \eta > 0. \tag{19}$$

Applying the Laplace transform to Eq. (18) and using the initial condition (19) we found that

$$\bar{F}_s(\eta,q) = \eta\sqrt{\frac{2}{\pi}}\frac{\nu}{\lambda q^2 + [1 + \lambda c]q + \left[\nu\eta^2 + \nu\frac{\phi}{k} + c\right]}\bar{U}(q), \tag{20}$$

where q is the transformed parameter while $\bar{F}_s(\eta,q)$ and $\bar{U}(q)$ are the Laplace transform of $F_s(\eta,t)$ and $U(t)$, respectively. Choosing $U(t) = UH(t)$, where $H(t)$ is Heaviside unit step function and U is the constant; we get the velocity field corresponding to the Rayleigh–Stokes problem.

In the case $\bar{U}(q) = \frac{U}{q}$ Eq. (20) takes the form

$$\bar{F}_s(\eta,q) = \frac{U\eta}{\lambda}\sqrt{\frac{2}{\pi}}\frac{1}{q}\left(\frac{\nu}{(q - r_1)(q - r_2)}\right), \tag{21}$$

and

$$r_1, r_2 = \frac{-[1 + \lambda c] \pm \sqrt{[1 + \lambda c]^2 - 4\lambda(\nu\eta^2 + \nu\frac{\phi}{k} + c)}}{2\lambda}.$$

Applying the inverse Laplace transform to (21), the solution can be expressed as

$$F_s(\eta,t) = U\sqrt{\frac{2}{\pi}}\left[\frac{\eta}{(\eta^2 + \frac{\phi}{k} + \frac{c}{\nu})}\left(1 - \frac{r_2 e^{r_1 t} - r_1 e^{r_2 t}}{r_2 - r_1}\right)\right], \quad t > 0. \tag{22}$$

Inversion of Fourier sine transform in (22) then gives

$$F(z,t) = UH(t)\left[e^{-\left(\sqrt{\frac{\phi}{k} + \frac{c}{\nu}}\right)z} - \frac{2}{\pi}\int_0^\infty \left(\frac{r_2 e^{r_1 t} - r_1 e^{r_2 t}}{r_2 - r_1}\right)\frac{\eta\sin(z\eta)}{(\eta^2 + \frac{\phi}{k} + \frac{c}{\nu})}d\eta\right]. \tag{23}$$

The velocity field (23) is in different form from that of [10], and obtained in another way. Of course, the velocity field (23) is obtained in accordance with the same method used to generate the results in [13–15].

The above expression (23) for hydrodynamic fluid (i.e. $B_0^2 = 0$) in a non-porous space (i.e. $\phi = 0$) is given by

$$F(z,t) = UH(t)\left[e^{-(1+i)\sqrt{\frac{\Omega}{\nu}}z} - \frac{2}{\pi}\int_0^\infty \left(\frac{r_4 e^{r_3 t} - r_3 e^{r_4 t}}{r_4 - r_3}\right)\frac{\eta\sin(z\eta)}{(\eta^2 + \frac{2i\Omega}{\nu})}d\eta\right], \tag{24}$$

where

$$r_3, r_4 = \frac{-[1 + 2i\lambda\Omega] \pm \sqrt{[1 + 2i\lambda\Omega]^2 - 4\lambda(\nu\eta^2 + 2i\Omega)}}{2\lambda}. \tag{25}$$

Putting $\Omega = 0$ into Eqs. (24) and (25), we obtain

$$F(z,t) = UH(t)\left[1 - \frac{2}{\pi}\int_0^\infty \left(\frac{r_6 e^{r_5 t} - r_5 e^{r_6 t}}{r_6 - r_5}\right)\frac{\eta\sin(z\eta)}{\eta}d\eta\right], \tag{26}$$

where

$$r_5, r_6 = \frac{-1 \pm \sqrt{1 - 4\lambda\nu\eta^2}}{2\lambda}. \tag{27}$$

The velocity field $F(z,t)$, given by Eq. (26), has been recently obtained by Fetecau et al. [13, Eq. (23)].

The result (23) for a magnetohydrodynamic viscous fluid, where $\lambda = 0$ in a porous space is

$$F(z,t) = UH(t)\left[e^{-\left(\sqrt{\frac{\phi}{k} + \frac{c}{\nu}}\right)z} - \frac{2}{\pi}\int_0^\infty \frac{\eta e^{-\left(\nu\eta^2 + \nu\frac{\phi}{k} + c\right)t}}{(\eta^2 + \frac{\phi}{k} + \frac{c}{\nu})}\sin(z\eta)\,d\eta\right], \tag{28}$$