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New high accuracy super stable alternating direction implicit methods for two and three dimensional hyperbolic damped wave equations

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ABSTRACT

In this paper, we report new three level implicit super stable methods of order two in time and four in space for the solution of hyperbolic damped wave equations in one, two and three space dimensions subject to given appropriate initial and Dirichlet boundary conditions. We use uniform grid points both in time and space directions. Our methods behave like fourth order accurate, when grid size in time-direction is directly proportional to the square of grid size in space-direction. The proposed methods are super stable. The resulting system of algebraic equations is solved by the Gauss elimination method. We discuss new alternating direction implicit (ADI) methods for two and three dimensional problems. Numerical results and the graphical representation of numerical solution are presented to illustrate the accuracy of the proposed methods.

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1. Introduction

We consider the damped wave equation

$$u_{tt} + 2\alpha u_t = a^2 u_{xx} + f(x,t), \quad 0 < x < \pi, \ 0 < t < T \tag{1}$$

where 'a' is the propagation speed of the wave, 'α' is a small positive damping constant. The right-hand side function $f(x,t)$ is an arbitrary external forcing function. For simplicity, we assume that the length of the string is one and the constant $a^2 = 1$. Mickens and Jordan [1,2] have studied a new non-standard finite difference scheme for the positive solution of damped wave equation. Mohanty et al. [3–12] have developed high accuracy methods for the solution of multi-dimensional nonlinear hyperbolic equations, in which, they have shown that the schemes for linear hyperbolic equations are conditionally stable. Later, Mohanty et al. [13–18] have discussed lower order unconditionally stable schemes for the solution of multi-dimensional Telegraphic equations. Although Eq. (1) is a particular case of Telegraphic equation, the unknown parameters involved in the schemes discussed in [13–18] are dependent on the grid sizes and mesh ratio parameter. Other lower order methods for multi-dimensional Telegraphic equations are discussed in the literature [19–32]. In this paper, we discuss new three level implicit super stable methods of order two in time and four in space for the solution of one, two and three space dimensional damped wave equation. In next section, we derive the super stable method for one space dimensional damped wave equation and discuss the stability analysis. In this method, we use three uniform spatial grid points at each time level. In Section 3, we discuss a new alternating direction implicit (ADI) super stable method, and the stability analysis for two dimensional problems. In Section 4, we extend our technique, and present stability analysis and ADI super stable method for three dimensional problems. In Section 5, we solve multidimensional damped wave equation using the proposed methods and compare the results with the results of other existing methods. Concluding remarks are given in Section 6.

2. Super stable method for one dimensional damped wave equation

For simplicity, we consider the damped wave equation

$$u_{tt} + 2\alpha u_t = u_{xx} + f(x,t), \quad \alpha > 0 \tag{2}$$

over a region $\Omega = \{(x,t) | 0 < x < 1, \ t > 0\}$, with the initial conditions

$$u(x,0) = a_0(x), \quad u_t(x,0) = a_1(x), \ 0 \leqslant x \leqslant 1, \tag{3}$$

and boundary conditions

$$u(0,t) = b_0(t), \quad u(1,t) = b_1(t), \quad t > 0. \tag{4}$$

We assume that $a_0(x)$, $a_1(x)$, and their derivatives are continuous functions of x. For the numerical solution of the above initial boundary value problem, we divide the interval [0,1] into $(N+1)$ subintervals each of width $h > 0$, so that $(N+1)h = 1$. Let $\tau > 0$ be the step size in the time direction. The grid points are given by $(x_l, t_j) = (lh, j\tau)$; $l = 0(1)N+1$, $j = 1,2,3,\ldots$. Let U_l^j be the exact solution

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value of $u(x,t)$ at the grid point (x_l,t_j) and u_l^j be the approximate value of U_l^j. Throughout the paper, we denote $a = \alpha^2\tau^2$ and $\lambda = (\tau/h) > 0$ be the mesh ratio parameter.

Applying the method discussed in [3], a three level implicit method of $O(\tau^4 + \tau^2h^2 + h^4)$ for the differential equation (2) may be written as

$$\left(1+\frac{a}{3}\right)\delta_t^2 u_l^j + \sqrt{a}(2\mu_t\delta_t)u_l^j + \frac{\sqrt{a}}{12}(1-\lambda^2)(\delta_x^2 2\mu_t\delta_t)u_l^j - \lambda^2\delta_x^2 u_l^j + \left(\frac{1-\lambda^2}{12}\right)\delta_x^2\delta_t^2 u_l^j = F_{10} \quad (5)$$

where $\delta_t u_l^j = u_l^{j+\frac{1}{2}} - u_l^{j-\frac{1}{2}}$ and $\mu_t u_l^j = \frac{1}{2}\left(u_l^{j+\frac{1}{2}} + u_l^{j-\frac{1}{2}}\right)$ are central and averaging operators with respect to t-direction etc., $f_l^j = f(x_l, t_j)$ and

$$F_{10} = \frac{\tau^2}{12}\left[f_{l+1}^j + f_{l-1}^j + (1+\sqrt{a})f_l^{j+1} + (1-\sqrt{a})f_l^{j-1} + 8f_l^j\right]$$

Now, we discuss the stability of the scheme (5). The exact solution value U_l^j satisfies

$$\left(1+\frac{a}{3}\right)\delta_t^2 U_l^j + \sqrt{a}(2\mu_t\delta_t)U_l^j + \frac{\sqrt{a}}{12}(1-\lambda^2)(\delta_x^2 2\mu_t\delta_t)U_l^j - \lambda^2\delta_x^2 U_l^j + \left(\frac{1-\lambda^2}{12}\right)\delta_x^2\delta_t^2 U_l^j = F_{10} + O(\tau^6 + \tau^4h^2 + \tau^2h^4) \quad (6)$$

We assume that there exists an error $\varepsilon_l^j = u_l^j - U_l^j$ at the grid point (x_l,t_j). Subtracting (6) from (5), we obtain the corresponding error equation

$$\left(1+\frac{a}{3}\right)\delta_t^2 \varepsilon_l^j + \sqrt{a}(2\mu_t\delta_t)\varepsilon_l^j + \frac{\sqrt{a}}{12}(1-\lambda^2)(\delta_x^2 2\mu_t\delta_t)\varepsilon_l^j - \lambda^2\delta_x^2 \varepsilon_l^j + \left(\frac{1-\lambda^2}{12}\right)\delta_x^2\delta_t^2 \varepsilon_l^j = O(\tau^6 + \tau^4h^2 + \tau^2h^4) \quad (7)$$

For stability, we put $\varepsilon_l^j = \xi^j e^{i\theta l}$ in the homogeneous part of the error equation (7), we get the characteristic equation

$$P\xi^2 + Q\xi + R = 0 \quad (8)$$

where

$$P = 1 + \frac{a}{3} + \sqrt{a} - (1+\sqrt{a})\left(\frac{1-\lambda^2}{3}\right)\sin^2\left(\frac{\theta}{2}\right),$$

$$Q = -2 - \frac{2a}{3} + \frac{2(1-\lambda^2)}{3}\sin^2\left(\frac{\theta}{2}\right) + 4\lambda^2\sin^2\left(\frac{\theta}{2}\right),$$

$$R = 1 + \frac{a}{3} - \sqrt{a} - (1-\sqrt{a})\left(\frac{1-\lambda^2}{3}\right)\sin^2\left(\frac{\theta}{2}\right).$$

Using the transformation $\xi = \frac{1+z}{1-z}$, the characteristic equation (8) reduces to

$$(P - Q + R)z^2 + 2(P - R)z + (P + Q + R) = 0 \quad (9)$$

The necessary and sufficient condition for $|\xi| < 1$ is that

$$P + Q + R > 0, \quad P - R > 0, \quad P - Q + R > 0.$$

Thus for stability, we must have the conditions

(i) $P + Q + R = 4\lambda^2\sin^2\left(\frac{\theta}{2}\right) > 0$ for all θ except $\theta = 0$ and 2π.
(ii) $P - R = \frac{2\sqrt{a}}{3}\left(2 + \cos^2\left(\frac{\theta}{2}\right) + \lambda^2\sin^2\left(\frac{\theta}{2}\right)\right) > 0$ for all variable angle θ.
(iii) $P - Q + R = \frac{4a}{3} + 4 - \frac{4}{3}\sin^2\frac{\theta}{2} - \frac{8\lambda^2}{3}\sin^2\left(\frac{\theta}{2}\right) > 0$, if

$$4 - \frac{4}{3}\sin^2\frac{\theta}{2} - \frac{8\lambda^2}{3}\sin^2\left(\frac{\theta}{2}\right) > 0,$$

or,

$$\frac{2\lambda^2}{3}\sin^2\left(\frac{\theta}{2}\right) < 1 - \frac{1}{3}\sin^2\frac{\theta}{2}. \quad (10)$$

The inequality (10) holds good, if $\max\left[\frac{2\lambda^2}{3}\sin^2\left(\frac{\theta}{2}\right)\right] < \min\left[1 - \frac{1}{3}\sin^2\frac{\theta}{2}\right]$, that is, $0 < \lambda^2 < 1$. Thus the scheme (5) is stable, if $0 < \lambda^2 < 1$.

In order to obtain a stable scheme with extended stability range, we follow the ideas given by Chawla [33]. We may re-write (5) in a modified form

$$\left(1+\frac{a}{3}\right)\delta_t^2 u_l^j + \sqrt{a}(2\mu_t\delta_t)u_l^j + \frac{\sqrt{a}}{12}(1-\lambda^2)(\delta_x^2 2\mu_t\delta_t)u_l^j - \lambda^2\delta_x^2 u_l^j + \left(\frac{1-\lambda^2}{12} - \gamma\lambda^2\right)\delta_x^2\delta_t^2 u_l^j = F_{10} \quad (11)$$

where $\gamma > 0$ is a free parameter to be determined. Although the additional term is of $O(\tau^4)$, it enables us to determine the values of parameter γ for which the method is super stable. For $\tau \propto h^2$, the method (11) behaves like a fourth order method. The exact solution value U_l^j satisfies

$$\left(1+\frac{a}{3}\right)\delta_t^2 U_l^j + \sqrt{a}(2\mu_t\delta_t)U_l^j + \frac{\sqrt{a}}{12}(1-\lambda^2)(\delta_x^2 2\mu_t\delta_t)U_l^j - \lambda^2\delta_x^2 U_l^j + \left(\frac{1-\lambda^2}{12} - \gamma\lambda^2\right)\delta_x^2\delta_t^2 U_l^j = F_{10} + O(\tau^4 + \tau^4h^2 + \tau^2h^4) \quad (12)$$

Subtracting (12) from (11), we obtain the error equation

$$\left(1+\frac{a}{3}\right)\delta_t^2 \varepsilon_l^j + \sqrt{a}(2\mu_t\delta_t)\varepsilon_l^j + \frac{\sqrt{a}}{12}(1-\lambda^2)(\delta_x^2 2\mu_t\delta_t)\varepsilon_l^j - \lambda^2\delta_x^2 \varepsilon_l^j + \left(\frac{1-\lambda^2}{12} - \gamma\lambda^2\right)\delta_x^2\delta_t^2 \varepsilon_l^j = O(\tau^4 + \tau^4h^2 + \tau^2h^4) \quad (13)$$

For stability, we put $\varepsilon_l^j = \xi^j e^{i\theta l}$ in the homogeneous part of the error equation (13), we get the characteristic equation

$$A\xi^2 + B\xi + C = 0 \quad (14)$$

where

$$A = 1 + \frac{a}{3} + \sqrt{a} - (1+\sqrt{a})\left(\frac{1-\lambda^2}{3}\right)\sin^2\left(\frac{\theta}{2}\right) + 4\gamma\lambda^2\sin^2\left(\frac{\theta}{2}\right),$$

$$B = -2 - \frac{2a}{3} + \frac{2(1-\lambda^2)}{3}\sin^2\left(\frac{\theta}{2}\right) + (1-2\gamma)4\lambda^2\sin^2\left(\frac{\theta}{2}\right),$$

$$C = 1 + \frac{a}{3} - \sqrt{a} - (1-\sqrt{a})\left(\frac{1-\lambda^2}{3}\right)\sin^2\left(\frac{\theta}{2}\right) + 4\gamma\lambda^2\sin^2\left(\frac{\theta}{2}\right).$$

Using the transformation $\xi = (1+z)/(1-z)$, the characteristic equation (14) reduces to

$$(A - B + C)z^2 + 2(A - C)z + (A + B + C) = 0 \quad (15)$$

The necessary and sufficient condition for $|\xi| < 1$ is that

$$A + B + C > 0, \quad A - C > 0, \quad A - B + C > 0 \quad (16)$$

Thus for stability, we must have the conditions

(i) $A + B + C = 4\lambda^2\sin^2\left(\frac{\theta}{2}\right) > 0$ for all θ except $\theta = 0$ and 2π. We can treat this separately.
(ii) $A - C = \frac{2\sqrt{a}}{3}\left(2 + \cos^2\left(\frac{\theta}{2}\right) + \lambda^2\sin^2\left(\frac{\theta}{2}\right)\right) > 0$ for all variable angle θ.
(iii) $A - B + C = \left(4 - \frac{4}{3}\sin^2\frac{\theta}{2}\right) + \frac{4a}{3} + \left(16\gamma - \frac{8}{3}\right)\lambda^2\sin^2\left(\frac{\theta}{2}\right) > 0$, if $\gamma \geqslant \frac{1}{6}$.
A method will be called *superstable*, if the period of stability is $(0,\infty)$.

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